Comm. Korean Math. Soc. **10** (1995), No. 4, pp. 881–887

ON CERTAIN MAXIMAL OPERATORS BEING A_1 WEIGHTS

Choon-Serk Suh

Abstract. Let f be a measurable function on the unit ball B in C^n, then we define a maximal function $M_p(f)$, $1 \le p < \infty$, by

$$M_p(f)(\zeta) = \sup_{\delta>0} \left(\frac{1}{\sigma(\beta(\zeta,\delta))} \int_{T(\beta(\zeta,\delta))} |f(z)|^p \frac{d\nu(z)}{(1-|z|)^n} \right)^{1/p}$$

where σ denotes the surface area measure on S, the boundary of B, and $T(\beta(\zeta,\delta))$ denotes the tent over the ball $\beta(\zeta,\delta)$. We prove that the maximal operator M_p belongs to the Muckenhoupt class A_1.

1. Introduction

Let S be the boundary of the unit ball B in the complex n-space C^n. Then we shall consider a maximal operator M_p defined on S; let f be a measurable function f on B and $1 \le p < \infty$, then we define a maximal function $M_p(f)$, for $\zeta \in S$, by

$$M_p(f)(\zeta) = \sup_{\delta>0} \left(\frac{1}{\sigma(\beta(\zeta,\delta))} \int_{T(\beta(\zeta,\delta))} |f(z)|^p \frac{d\nu(z)}{(1-|z|)^n} \right)^{1/p}$$

Here σ denotes the surface area measure on S, and $T(\beta(\zeta,\delta))$ denotes the tent over the ball $\beta(\zeta,\delta)$.

In this paper we study that the maximal operator M_p belongs to the Muckenhoupt class A_1. To prove this we need to prove that this operator M_p is of weak type (p,p) for $1 \le p < \infty$.

Received March 13, 1995. Revised August 1, 1995.

1991 AMS Subject Classification: 42B20, 42B25.

Key words: maximal operator, A_1 condition, weak type (p,p), tent.

2. Preliminaries

Let $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$ be two vectors in the complex n-space C^n. Then the *inner product* $\langle z, w\rangle$ of z and w will be given by

$$\langle z, w\rangle = \sum_{i=1}^{n} z_i \overline{w}_i,$$

where $\overline{w}_i$ is the complex conjugate of w_i, and the associated *norm* $|z|$ of z will be denoted by

$$|z| = \langle z, z\rangle^{1/2} = \left(\sum_{i=1}^{n} |z_i|^2\right)^{1/2}.$$

The *open unit ball* of C^n will be denoted by B, i.e., $B = \{z \in C^n : |z| < 1\}$. The *boundary* of B is the sphere S, that is, the set $S = \{z \in C^n : |z| = 1\}$.

Throughout this paper, we let σ denote the surface area measure on S. Note that the Lebesque measure ν and the measure σ are related by the formula

$$\int_{C^n} f(z) d\nu(z) = 2n \int_0^\infty r^{2n-1} dr \int_S f(r\zeta) d\sigma(\zeta). \tag{2.1}$$

For (2.1) see [5, p.13].

For $\zeta, \eta \in S$ and $\delta > 0$, let

$$\rho(\zeta, \eta) = |1 - \langle \zeta, \eta\rangle|$$

and

$$\beta(\zeta, \delta) = \{\eta \in S : |1 - \langle \zeta, \eta\rangle| < \delta\}.$$

Then it is easy to check that ρ defines a pseudo-metric on S and the triple $(S, \rho, d\sigma)$ becomes a space of homogeneous type (see [1]).

For $\alpha > 1$ and $\zeta \in S$, the set

$$\mathcal{A}_\alpha(\zeta) = \{z \in B : |1 - \langle z, \zeta\rangle| < \alpha(1 - |z|)\}.$$

is called an *admissible approach region.* This terminology is due to Korányi [3].

For a closed subset $F \subset S$, we define the *tent* over an open subset $O = F^c$, as

$$T(O) = B \setminus \bigcup_{\zeta \in F} \mathcal{A}_4(\zeta).$$

For a measurable function f defined on B and $1 \leq p < \infty$, we define a maximal function $M_p(f)$, for $\zeta \in S$, by

$$(2.2) \qquad M_p(f)(\zeta) = \sup_{\delta > 0} \left(\frac{1}{\sigma(\beta(\zeta, \delta))} \int_{T(\beta(\zeta,\delta))} |f(z)|^p \frac{d\nu(z)}{(1-|z|)^n} \right)^{1/p}$$

As usual, throughout this paper C will denote a constant not necessarily the same at each occurrence.

3. Main result

LEMMA 1 (COVERING LEMMA, [1]). *Let E be contained in some ball in S. Let $\delta(\zeta)$ be a positive number for each $\zeta \in E$. Then there is a sequence of disjoint balls $\beta(\zeta_i, \delta(\zeta_i))$, $\zeta_i \in E$, such that*

$$E \subset \bigcup_i \beta(\zeta_i, 4K\delta(\zeta_i)),$$

where K is the constant in the triangle inequality. Furthermore, every $\zeta \in E$ is contained in some ball $\beta(\zeta_i, 4K\delta(\zeta_i))$ satisfying $\delta(\zeta) \leq 2\delta(\zeta_i)$.

PROPOSITION 2. *Let M_p be defined as in (2.2). Then the maximal operator M_p is of weak type (p,p) for $1 \leq p < \infty$.*

PROOF. Fix $\lambda > 0$, set

$$E_\lambda = \{\zeta \in S : M_p(f)(\zeta) > \lambda\}.$$

For each $\zeta \in E_\lambda$, let

$$\delta(\zeta) = \sup \left\{ \delta > 0 : \left(\frac{1}{\sigma(\beta(\zeta, \delta))} \int_{T(\beta(\zeta,\delta))} |f(z)|^p \frac{d\nu(z)}{(1-|z|)^n} \right)^{1/p} > \lambda \right\}.$$

Thus for each $\zeta \in E_\lambda$, we have $\delta(\zeta) > 0$ and

$$\frac{1}{\sigma(\beta(\zeta,\delta(\zeta)))}\int_{T(\beta(\zeta,\delta(\zeta)))}|f(z)|^p\frac{d\nu(z)}{(1-|z|)^n} \geq \lambda^p. \tag{3.1}$$

Apply Lemma 1 to the balls $\beta(\zeta,\delta(\zeta))$ to obtain a sequence of disjoint balls $\beta(\zeta_i,\delta(\zeta_i))$, so that

$$E_\lambda \subset \bigcup_i \beta(\zeta_i, 4K\delta(\zeta_i)).$$

Then it follows from the doubling property of σ [1] and (3.1) that

$$\begin{aligned}
\sigma(E_\lambda) &\leq \sum_i \sigma(\beta(\zeta_i, 4K\delta(\zeta_i)))\\
&\leq C\sum_i \sigma(\beta(\zeta_i,\delta(\zeta_i)))\\
&\leq \frac{C}{\lambda^p}\sum_i\int_{T(\beta(\zeta_i,\delta(\zeta_i)))}|f(z)|^p\frac{d\nu(z)}{(1-|z|)^n}\\
&\leq \frac{C}{\lambda^p}\int_B |f(z)|^p\frac{d\nu(z)}{(1-|z|)^n}\\
&= C(\|f\|_p/\lambda)^p,
\end{aligned}$$

since the balls $\beta(\zeta_i,\delta(\zeta_i))$ are disjoint. Hence the maximal operator M_p is of weak type (p,p). The proof is therefore complete. #

LEMMA 3. *Let M_p be defined as in (2.2). Then there is a constant C such that*

$$M_p(f)(\xi) \leq CM_p(f)(\eta)$$

for any $\xi,\eta \in \beta(\zeta,\delta)$.

PROOF. Let $\xi,\eta \in \beta(\zeta,\delta)$. If $M_p(f)(\xi) \neq 0$, then clearly $\eta \in \beta(\xi, C\delta)$ with $C > 1$. Thus

$$\begin{aligned}
M_p(f)(\xi) &\leq \sup_{\delta>0}\frac{1}{\sigma(\beta(\xi,\delta))}\left(\int_{T(\beta(\xi,C\delta))}|f(z)|^p\frac{d\nu(z)}{(1-|z|)^n}\right)^{1/p}\\
&\leq CM_p(f)(\eta).
\end{aligned}$$

The proof is therefore complete. #

THEOREM 4. *Let M_p be defined as in (2.2) and $1 \le p < \infty$. Then the maximal operator M_p satisfies the A_1 condition [4]; more precisely, there is a constant C such that*

$$\frac{1}{\sigma(\beta(\zeta,\delta))}\int_{\beta(\zeta,\delta)} M_p(f)(\xi)d\sigma(\xi) \le C \inf_{\xi\in\beta(\zeta,\delta)} M_p(f)(\xi)$$

for all balls $\beta(\zeta,\delta)$ containing ξ.

PROOF. Let

$$M_1(u)(\zeta) = \sup_{\delta>0}\frac{1}{\sigma(\beta(\zeta,\delta))}\int_{T(\beta(\zeta,\delta))} u(z)\frac{d\nu(z)}{(1-|z|)^n},$$

where $u(z) = |f(z)|^p$. For any ball $\beta(\zeta,\delta)$ in S, decompose

$$u(z) = u_1(z) + u_2(z),$$

where

$$u_1(z) = u(z)\chi_{T(\beta(\zeta,C\delta))}(z) \quad \text{for } C > 1.$$

Since M_1 is of weak type $(1,1)$, by the Kolmogorov's inequality there is a constant C such that

(3.2)

$$\int_{\beta(\zeta,\delta)} M_1(u_1)(\xi)^{1/p}d\sigma(\xi) \le C[\sigma(\beta(\zeta,\delta))]^{1-1/p}\left(\int_B u_1(z)\frac{d\nu(z)}{(1-|z|)^n}\right)^{1/p}$$

Thus it follows from (3.2) that

$$\begin{aligned}\frac{1}{\sigma(\beta(\zeta,\delta))}\int_{\beta(\zeta,\delta)} &M_1(u_1)(\xi)^{1/p}d\sigma(\xi)\\ &\le C\left(\frac{1}{\sigma(\beta(\zeta,\delta))}\int_B u_1(z)\frac{d\nu(z)}{(1-|z|)^n}\right)^{1/p}\\ &\le C\left(\frac{1}{\sigma(\beta(\zeta,\delta))}\int_{T(\beta(\zeta,C\delta))} u(z)\frac{d\nu(z)}{(1-|z|)^n}\right)^{1/p}\\ &\le CM_1(u)(\xi)^{1/p}\end{aligned}$$

for any $\xi \in \beta(\zeta, \delta)$. Thus

$$\frac{1}{\sigma(\beta(\zeta,\delta))} \int_{\beta(\zeta,\delta)} M_1(u_1)(\xi)^{1/p} d\sigma(\xi) \leq C \inf_{\xi \in \beta(\zeta,\delta)} M_1(u)(\xi)^{1/p}. \tag{3.3}$$

On the other hand, it follows from Lemma 3 that

$$M_1(u_2)(\xi) \leq C M_1(u_2)(\eta) \tag{3.4}$$

for any $\xi, \eta \in \beta(\zeta, \delta)$. Thus (3.4) implies that

(3.5)

$$\begin{aligned} \frac{1}{\sigma(\beta(\zeta,\delta))} \int_{\beta(\zeta,\delta)} M_1(u_2)(\xi)^{1/p} d\sigma(\xi) &\leq C \inf_{\eta \in \beta(\zeta,\delta)} M_1(u_2)(\eta)^{1/p} \\ &\leq C \inf_{\eta \in \beta(\zeta,\delta)} M_1(u)(\eta)^{1/p}. \end{aligned}$$

Since

$$M_1(u)(\xi)^{1/p} \leq C \left(M_1(u_1)(\xi)^{1/p} + M_1(u_2)(\xi)^{1/p} \right),$$

it follows from (3.3) and (3.5) that

$$\begin{aligned} &\frac{1}{\sigma(\beta(\zeta,\delta))} \int_{\beta(\zeta,\delta)} M_1(u)(\xi)^{1/p} d\sigma(\xi) \\ &\leq \frac{C}{\sigma(\beta(\zeta,\delta))} \left(\int_{\beta(\zeta,\delta)} M_1(u_1)(\xi)^{1/p} d\sigma(\xi) + \int_{\beta(\zeta,\delta)} M_1(u_2)(\xi)^{1/p} d\sigma(\xi) \right) \\ &\leq C \inf_{\xi \in \beta(\zeta,\delta)} M_1(u)(\xi)^{1/p}. \end{aligned}$$

Thus $M_1(u)^{1/p} = M_p(f) \in A_1$. The proof is therefore complete. #

References

1. R. R. Coifman and G. Weiss, *Analyse harmonique non-commutative sur certains éspaces homogènes*, Lecture Notes in Math. **242** (1971), Springer-Verlag, Berlin.
2. L. Hörmander, *L^p estimates for (pluri) subharmonic function*, Math. Scand. **20** (1967), 65-78.

3. A. Korányi, *Harmonic functions in hermitian hyperbolic space*, Trans. Amer. Math. Soc. **135** (1969), 507-516.
4. B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. **165** (1972), 207-226.
5. W. Rudin, *Function Theory in the Unit Ball of C^n*, Grudlehren der Mathematischen Wissenschaften in Einzeldarstellungen, Princeton Univ. Press, Princeton, N. J., 1980.
6. E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, Princeton, N. J., 1970.

Department of Mathematics
Dongyang University of Technology
Youngju 751-800, Korea