Comm. Korean Math. Soc. 10 (1995), No. 4, pp. 867–874

LINEARLY INVARIANT FUNCTIONS

TAI SUNG SONG

ABSTRACT. Linear invariance is closely related to the concept of uniform local univalence. We give a geometric proof that a holomorphic locally univalent function defined on the open unit disk is linearly invariant if and only if it is uniformly locally univalent.

1. Introduction

A holomorphic locally univalent function f defined on the open unit disk D is called *linearly invariant* if

$$\|f\|_L = \sup\left\{\left|\left(1-|z|^2\right)\frac{f''(z)}{2f'(z)} - \bar{z} : z \in D\right|\right\} < \infty.$$

Let $L(f,z) = \left|\left(1-|z|^2\right) f''(z)/2f'(z) - \bar{z}\right|$, and let $T \in Aut(D)$, the group of conformal automorphisms of D. Then

$$L(f\circ T, z) = \left|\frac{1}{2}\left(1-|z|^2\right)\left[\frac{f''(T(z))}{f'(T(z))}T'(z) + \frac{T''(z)}{T'(z)}\right] - \bar{z}\right|$$
$$= \left|\frac{T'(z)}{|T'(z)|}\right|\left|\frac{1}{2}\left(1-|z|^2\right)|T'(z)|\frac{f''(T(z))}{f'(T(z))} + \left[\frac{1}{2}\left(1-|z|^2\right)\frac{T''(z)}{T'(z)} - \bar{z}\right]\frac{|T'(z)|}{T'(z)}\right|$$
$$= \left|\frac{1}{2}\left[1-|T(z)|^2\right]\frac{f''(T(z))}{f'(T(z))} - \overline{T(z)}\right| = L(f, T(z)).$$

This shows that the quantity $\|f\|_L$ is invariant under the group $Aut(D)$ of conformal automorphisms of D : if T is a conformal automorphism of D, then $\|f\circ T\|_L = \|f\|_L$. By calculation, $\|f\|_L = 1$ if f is a conformal

Received February 25, 1995. Revised September 16, 1995.
1991 AMS Subject Classification: 30C55, 30D45.
Key words: hyperbolic metric, linear invariance, uniform local univalence.
The Present Studies were Supported by the Basic Science Research Institute Program, Ministry of Education, Project No. BSRI-94-1411.

automorphism of D. In fact, $\|f\|_L = 1$ if and only if f is a convex univalent function [7].

Linear invariance is closely related to the concept of uniform local univalence. The notion of uniform local univalence is defined relative to hyperbolic geometry on D. For a general discussion of hyperbolic geometry, see [2], [4] and [5]. The hyperbolic distance function on D induced by the hyperbolic metric $\lambda_D(z)\,|dz| = 2\,|dz| / \left(1 - |z|^2\right)$ is

$$d_h(a,b) = 2\tanh^{-1}\left|\frac{a-b}{1-a\bar{b}}\right|.$$

The hyperbolic disk in D with center $a \in D$ and hyperbolic radius ρ, $0 < \rho \leq \infty$, is defined by

$$D_h(a,\rho) = \{z \in D : d_h(a,z) < \rho\}.$$

We let $D(a,r)$ denote the euclidean disk with center a and radius r. The hyperbolic disk $D_h(a,\rho)$ is a euclidean disk $D(c,r)$, where

$$c = \frac{1 - \left(\tanh\frac{\rho}{2}\right)^2}{1 - \left(\tanh\frac{\rho}{2}\right)^2 |a|^2}\, a,$$

$$r = \tanh\frac{\rho}{2}\,\frac{1 - |a|^2}{1 - \left(\tanh\frac{\rho}{2}\right)^2 |a|^2}.$$

In particular, we have $D_h(0,\rho) = D\left(0, \tanh\frac{\rho}{2}\right)$. Suppose f is a holomorphic function on D. For $z \in D$, let $\rho(z,f)$ be the hyperbolic radius of the largest hyperbolic disk in D centered at z in which f is univalent. Set

$$\rho(f) = \inf\{\rho(z,f) : z \in D\}.$$

A function f is called *uniformly locally univalent* (in the hyperbolic sense) in D if $\rho(f) > 0$.

In this paper we investigate some properties of linearly invariant functions. In Section 2 we give an upper bound for the linear invariant norm. Ma and Minda [6] extended the notions of linear invariance and uniform local univalence to arbitrary hyperbolic regions and proved that two notions are equivalent. In Section 3 we give a new proof of this result in the special case of the open unit disk D by using elementary ideas from hyperbolic geometry on D.

2. Linear invariant norm

Let $f \in S$, the class of normalized univalent functions in D. Then

$$\left|\frac{f''(z)}{f'(z)} - \frac{2\bar{z}}{1-|z|^2}\right| \leq \frac{4}{1-|z|^2}, \quad z \in D, \tag{1}$$

and hence

$$\left|\frac{f''(0)}{f'(0)}\right| \leq 4. \tag{2}$$

See, for example [1, p.84]. Although, (1) and (2) are stated for the class S, they are valid for any holomorphic univalent function in D since the expression f''/f' is unchanged if f is replaced by $af+b$, $a,b \in \mathbf{C}$, $a \neq 0$.

THEOREM 1. *Suppose that f is a holomorphic locally univalent function of the open unit disk D into itself. Then*

$$1 \leq \|f\|_L \leq \frac{2}{\tanh\left[\rho(f)/2\right]}.$$

PROOF. The inequality $1 \leq \|f\|_L$ is well known. We include a proof for the convenience. Let $\alpha = \|f\|_L$. If $\alpha = \infty$, we are done. Suppose $\alpha < \infty$. If $|z| < r < 1$, then

$$\left|\frac{\partial}{\partial r}\log\left[\left(1-r^2\right)f'\left(re^{it}\right)\right]\right| = \left|\frac{-2r}{1-r^2} + \frac{f''\left(re^{it}\right)}{f'\left(re^{it}\right)}e^{it}\right|$$

$$= \frac{2}{1-r^2}\left|\left(1-r^2\right)\frac{f''\left(re^{it}\right)}{2f'\left(re^{it}\right)} - re^{-it}\right|.$$

and hence

$$\left|\frac{\partial}{\partial r}\log\left[\left(1-r^2\right)f'\left(re^{it}\right)\right]\right|\leq\frac{2\alpha}{1-r^2}. \tag{3}$$

Integration of (3) along a radius yields

$$\left|\log\left[\left(1-|z|^2\right)f'(z)\right]\right|\leq\alpha\log\frac{1+|z|}{1-|z|}.$$

Since $|Re\ w|\leq|w|$, it follows that

$$\left[\frac{1-|z|}{1+|z|}\right]^{\alpha}\leq\left(1-|z|^2\right)|f'(z)|\leq\left[\frac{1+|z|}{1-|z|}\right]^{\alpha}$$

or

$$\frac{(1-|z|)^{\alpha-1}}{(1+|z|)^{\alpha+1}}\leq|f'(z)|\leq\frac{(1+|z|)^{\alpha-1}}{(1-|z|)^{\alpha+1}}. \tag{4}$$

If $\alpha<1$, then it follows from (4) that $|f'(z)|\to\infty$ as $|z|\to 1$, contrary to the minimum principle applied to f'. Thus, we obtain the inequality $1\leq\|f\|_L$. Next, we establish the upper bound. We may assume that $\rho(f)>0$. For each $a\in D$, let $g(z)=(f\circ T)(z)$, where $T(z)=(z+a)/(1+\overline{a}z)$. Then g is univalent in each hyperbolic disk of radius $\rho(f)$. In particular, g is univalent in the euclidean disk centered at 0 with radius $r=\tanh[\rho(f)/2]$. Let $h(z)=g(rz)/r$. Then h is univalent in D. So, by (2), $|h''(0)|\leq 4|h'(0)|$. Therefore

$$L(f,a)=L(f,T(0))=L(f\circ T,0)=L(g,0)$$

$$=\left|\frac{g''(0)}{2g'(0)}\right|=\frac{1}{2r}\frac{|h''(0)|}{|h'(0)|}\leq\frac{2}{r}.$$

This yields

$$\|f\|_L\leq\frac{2}{r}=\frac{2}{\tanh[\rho(f)/2]}.$$

3. Linear invariance and uniform local univalence

Let $\delta_D(z) = dist(z, \partial D)$; this is the radius of the largest disk in D with center z. Note that $\delta_D(z) = 1 - |z|$. Becker [2] proved that if f is holomorphic and locally univalent in D, and if

$$\left(1 - |z|^2\right) \left|\frac{f''(z)}{f'(z)}\right| \leq 1$$

for all z in D, then f is univalent in D. The following result is a slight modification of Becker's univalence criterion.

LEMMA 2. *Suppose f is holomorphic and locally univalent in D. Let $a \in D$ and $\delta = \delta_D(a)$. If*

$$(\delta - |z - a|) \left|\frac{f''(z)}{f'(z)}\right| \leq \frac{M}{2}$$

for all $z \in D(a, \delta)$, where $M \geq 2$, then f is univalent in $D(a, \delta/M)$.

PROOF. For $z \in D$, let $g(z) = (f \circ h)(z)$, where $w = h(z) = \delta z/M + a$. Then

$$g'(z) = \frac{\delta}{M} f'\left(\frac{\delta}{M} z + a\right),$$

$$g''(z) = \left(\frac{\delta}{M}\right)^2 f''\left(\frac{\delta}{M} z + a\right),$$

so that

$$\begin{aligned}
\left(1 - |z|^2\right) \left|\frac{g''(z)}{g'(z)}\right| &= \left[1 - \left(\tfrac{M}{\delta}\right)^2 |w - a|^2\right] \tfrac{\delta}{M} \left|\frac{f''(w)}{f'(w)}\right| \\
&= \tfrac{1}{M\delta} \left[\delta^2 - M^2 |w - a|^2\right] \left|\frac{f''(w)}{f'(w)}\right| \\
&\leq \tfrac{1}{M\delta} (\delta - |w - a|)(\delta + |w - a|) \left|\frac{f''(w)}{f'(w)}\right| \\
&\leq \tfrac{1}{M\delta} 2\delta (\delta - |w - a|) \left|\frac{f''(w)}{f'(w)}\right| \\
&\leq \tfrac{2}{M} \tfrac{M}{2} = 1
\end{aligned}$$

for all z in D. Hence, by Becker's univalence criterion, g is univalent in D, so $f = g \circ h^{-1}$ is univalent in $D(a, \delta/M)$.

Lemma 3. *Suppose f is holomorphic and locally univalent in D. If f is linearly invariant, then there exists $r > 0$ such that f is univalent in $D(a, r\delta(a))$ for each a in D.*

Proof. Let $\delta = \delta(a) = 1 - |a|$. Since f is linearly invariant, it follows from Theorem 1 that there exists $M \geq 1$ such that

$$\left|\left(1 - |z|^2\right) \frac{f''(z)}{2f'(z)} - \overline{z}\right| \leq M$$

for all z in D. This implies that

$$(1 - |z|) \left|\frac{f''(z)}{f'(z)}\right| \leq \left(1 - |z|^2\right) \left|\frac{f''(z)}{f'(z)}\right| \leq 2(M + 1)$$

for all z in D. If $z \in D(a, \delta)$, then

$$(\delta - |z - a|) \left|\frac{f''(z)}{f'(z)}\right| \leq (1 - |z|) \left|\frac{f''(z)}{f'(z)}\right| \leq 2(M + 1).$$

Let $r = 1/(M + 1)$. Since $M + 1 \geq 2$, it follows from Lemma 2 that f is univalent in $D(a, r\delta(a))$.

Lemma 4. *Let $D_h(a, \rho)$ be a hyperbolic disk in D, and let $r = \tanh(\rho/2)$. Then*

$$\frac{rz + a}{1 + \overline{a}rz} \in D_h(a, \rho)$$

for all z in D.

Proof. We have

$$d_h\left(\frac{rz + a}{1 + \overline{a}rz}, a\right) = 2 \tanh^{-1} \frac{\left|a - \frac{rz + a}{1 + \overline{a}rz}\right|}{\left|1 - a\frac{r\overline{z} + \overline{a}}{1 + ar\overline{z}}\right|}$$

$$= 2 \tanh^{-1} r|z| < 2 \tanh^{-1} r = \rho.$$

This completes the proof.

We now show that the notion of linear invariance is equivalent to the notion of uniform local univalence.

THEOREM 5. *A holomorphic locally univalent function f defined on the open unit disk D is linearly invariant if and only if it is uniformly locally univalent in the hyperbolic sense.*

PROOF. First, suppose f is linearly invariant. Then, by Lemma 3, there exists $r > 0$ such that f is univalent in $D(a, r\delta(a))$ for each a in D. Let $c \in D$ and let $s \in (0,1)$. Then f is univalent in the euclidean disk

$$D\left(\frac{1-s^2}{1-s^2|c|^2}c, r\left[1-\frac{1-s^2}{1-s^2|c|^2}|c|\right]\right)$$

$$= D\left(\frac{1-s^2}{1-s^2|c|^2}c, \frac{r\left[1-|c|-s^2|c|^2+s^2|c|\right]}{1-s^2|c|^2}\right).$$

We note that if $s < \frac{1}{2}r$, then

$$s\left(1-|c|^2\right) < 2s(1-|c|)$$

$$< r\left[1-|c|-s^2|c|^2+s^2|c|\right].$$

This implies that f is univalent in the euclidean disk

$$D\left(\frac{1-s^2}{1-s^2|c|^2}c, \frac{s\left(1-|c|^2\right)}{1-s^2|c|^2}\right).$$

Let $\rho = 2\tanh^{-1}s$. Then f is univalent in $D_h(c, \rho)$.

Next, suppose that there exists $\rho > 0$ such that f is univalent in each hyperbolic disk of radius ρ. Let $c \in D$ and $r = \tanh(\rho/2)$. Then, by Lemma 4,

$$\frac{rz+c}{1+\overline{c}rz} \in D_h(c, \rho)$$

for all z in D. Hence, the function g defined by

$$g(z) = \frac{1}{r}\frac{f\left(\frac{rz+c}{1+\overline{c}rz}\right)-f(c)}{\left(1-|c|^2\right)f'(c)}$$

is univalent in D. We have

$$g(0)=0,\ g'(0)=1,$$

$$g''(0)=r\left(1-|c|^2\right)\frac{f''(c)}{f'(c)}-2\overline{c}r.$$

Therefore, by Bieberbach-deBrange Theorem, we have

$$\left|\left(1-|c|^2\right)\frac{f''(c)}{2f'(c)}-\overline{c}\right|=\left|\frac{g''(0)}{2r}\right|\leq\frac{2}{r},\ c\in D.$$

This shows that f is linearly invariant.

References

1. L. V. Ahlfors, *Conformal invariants. Topics in geometric function theory*, McGraw-Hill, New York, 1973.
2. A. F. Beardon, *The geometry of discrete groups*, Springer-Verlag, New York, 1983.
3. J. Becker, *Löwnersche Differentialgleichung und quasikonform fortsetzbare schlichte Funktionen*, J. Reine Angew. Math **255** (1972), 23-43.
4. H. M. Farkas and I. Kra, *Riemann surfaces*, Springer-Verlag, New York, 1980.
5. G. M. Goluzin, *Geometric theory of functions of a complex variable, Translations of Mathematical Monographs*, **26**, Amer. Math. Soc., Providence, 1969.
6. W. Ma and D. Minda, *Linear invariance and uniform local univalence*, Complex Variables Theory Appl. **16** (1991), 9-19.
7. Ch. Pommerenke, *Linear-invariante Familien analytischer Funktionen I*, Math. Ann. **155** (1964), 108-154.

Department of Mathematics Education
Pusan National University
Pusan 609-735, Korea