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REFLEXIVITY OF NORMED ALMOST LINEAR SPACES

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ABSTRACT. We prove that if a nals X is reflexive, then $X = W_X + V_X$. We prove also that if an als X has a finite basis, then $X = W_X + V_X$ if and only if X is reflexive.

1. Introduction

G. Godini[2,3,4] introduced the almost linear space(*als*), a concept which generalizes linear space(*ls*). In this paper, we introduce the algebraic dual space and algebraic double dual space of the *als* X, and define algebraic reflexivity of the *als* X. In general, the *als* X is not embeddable into its algebraic double dual space whereas it is true for linear spaces.

Using the concept of basis of an *als* introduced by Godini[2], we obtain the following results :

(1) If an *als* X has a basis, then X is embeddable in the algebraic double dual space $X^{\#\#}$.
(2) If an *als* X is algebraically reflexive then $X = W_X + V_X$.
(3) If an *als* X has a finite basis and $X = W_X + V_X$, then X is algebraically reflexive. Hence we have,
(4) If an *als* X has a finite basis, then $X = W_X + V_X$ if and only if X is algebraically reflexive.

In Section 4, we also study the normed almost linear space(*nals*) and show that the results described above are true for a *nals* X.
All spaces involved in this paper are over the real field $\mathbb{R}$.

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2. Preliminaries of ALS

An *almost linear space* (*als*) is a set X together with two mappings $s : X \times X \to X$ and $m : \mathbb{R} \times X \to X$ satisfying the conditions $(L_1)-(L_8)$ given below. For $x, y \in X$ and $\lambda \in \mathbb{R}$ we denote $s(x,y)$ by $x+y$ and $m(\lambda, x)$ by λx, when these will not lead to misunderstandings. Let $x, y, z \in X$ and $\lambda, \mu \in \mathbb{R}$. (L_1) $x+(y+z) = (x+y)+z$; (L_2) $x+y = y+x$; (L_3) There exists an element $0 \in X$ such that $x+0 = x$ for each $x \in X$; (L_4) $1x = x$; (L_5) $\lambda(x+y) = \lambda x + \lambda y$; (L_6) $0x = 0$; (L_7) $\lambda(\mu x) = (\lambda\mu)x$; (L_8) $(\lambda+\mu)x = \lambda x + \mu x$ for $\lambda \geq 0,\ \mu \geq 0$.

We denote $-1x$ by $-x$, if there is no confusion likely, and in the sequel $x - y$ means $x + (-y)$.

Note that $(\lambda + \mu)x = \lambda x + \mu x$ for every scalars $\lambda, \mu \in \mathbb{R}$ in a linear space, and $x - x$ need not be equal to zero for every x in an almost linear space.

A nonempty subset Y of an *als* X is called an *almost linear subspace* of X, if for each $y_1, y_2 \in Y$ and $\lambda \in \mathbb{R}$, $s(y_1, y_2) \in Y$ and $m(\lambda, y_1) \in Y$. An almost linear subspace Y of X is called a *linear subspace* of X if $s : Y \times Y \to Y$ and $m : \mathbb{R} \times Y \to Y$ satisfy all the axioms of a linear space.

For an *als* X we introduce the following two sets;

$$V_X = \{x \in X : x - x = 0\}, \tag{2-1}$$

$$W_X = \{x \in X : x = -x\}. \tag{2-2}$$

Then, we have the following properties: (1) The set V_X is a linear subspace of X, and it is the largest one. (2) The set W_X is an almost linear subspace of X and $W_X = \{x - x : x \in X\}$. (3) The *als* X is a linear space $\iff V_X = X \iff W_X = \{0\}$. (4) $V_X \cap W_X = \{0\}$.

A subset B of the *als* X is called a *basis* for X if for each $x \in X \setminus \{0\}$ there exist unique subsets $\{b_1, b_2, ..., b_n\} \subset B$, $\{\lambda_1, \lambda_2, ..., \lambda_n\} \subset \mathbb{R} \setminus \{0\}$ (n depending on x) such that $x = \sum_{i=1}^{n} \lambda_i b_i$, where $\lambda_i > 0$ for those

$b_i \notin V_X$. We shall call such a representation $x = \sum_{i=1}^{n} \lambda_i b_i$ the unique *positive representation* of x by the basis B.

In contrast to the case of a *ls*, there exists an *als* which have no basis.

EXAMPLES 2.1. These examples are from [2].

(1) Let $X = \{x \in \mathbb{R} : x \geq 0\}$. Define $s(x, y) = max\{x, y\}$ and $m(\lambda, x) = x$ for $\lambda \neq 0$, $m(0, x) = 0$. The element $0 \in X$ is $0 \in \mathbb{R}$. Then X is an *als*. We have $V_X = \{0\}$ and $W_X = X$. It is clear that X has no basis.

(2) Let $X = \{[a, b] \subset \mathbb{R} : a \leq b\}$. Define $s(A, B) = \{a + b : a \in A,\ b \in B\}$ and $m(\lambda, A) = \{\lambda a : a \in A\}$ for $A, B \in X$, $\lambda \in \mathbb{R}$. Then X is an *als*. We have $V_X = \{\{a\} \in X : a \in \mathbb{R}\}$ and $W_X = \{[-a, a] \in X : a \geq 0\}$. $X = W_X + V_X$ and $B = \{[-1, 1], \{1\}\}$ is a basis for the *als* $X = W_X + V_X$. Furthermore, $\{[-1, 1]\}$ is a basis for W_X and $\{\{1\}\}$ is a basis for V_X. Consider the almost linear subspace $Y = \{[a, b] \in X : a \leq 0,\ b \geq 0\}$ of X. $B_1 = \{[-1, 0], [0, 1]\}$ is a basis for Y. Note that $W_Y = \{[-a, a] : a \geq 0\}$, $V_Y = \{\{0\}\}$ and $Y \neq W_Y + V_Y$.

The following theorem will play an important role in our discussion:

THEOREM 2.2. *Suppose an als X has a basis B. Then,*

(1) [2; (2.8)] *There exists a basis B' of X with the property that for each $b' \in B' \setminus V_X$ we have $-b' \in B' \setminus V_X$. Moreover* $\mathrm{card}(B \setminus V_X) = \mathrm{card}(B' \setminus V_X)$. *We shall call such a basis a* **symmetric** *basis.*
(2) [2; (2.9)] *Let B' be a symmetric basis for an als X. Then $\{x - x : x \in B \setminus V_X\}$ is a basis for W_X.*
(3) [2; (2.10)] *There exist a norm $\|\cdot\|$ and a metric ρ on X for which X is a snals.*
(4) [2; (2.11)] (a) *The relations $x + y = x + z$, $x, y, z \in X$ imply that $y = z$;* (b) *The relations $w_1 + v_1 = w_2 + v_2$, $w_i \in W_X$, $v_i \in V_X$, $i = 1, 2$ imply that $w_1 = w_2$ and $v_1 = v_2$.*
(5) [2; (2.12)] *There exists a basis B'' of $W_X + V_X$ with the property that $B'' = B_1 \cup B_2$, where B_1 is a basis for W_X and B_2 is a basis for V_X.*

3. Algebraic reflexivity of ALS

Let X and Y be two almost linear spaces. A mapping $T : X \to Y$ is called a *linear operator* if $T(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 T(x_1) + \lambda_2 T(x_2)$ for all $\lambda_i \in \mathbb{R}$ and $x_i \in X, i = 1, 2$. An *isomorphism* T of an *als* X onto an *als* Y is a bijective mapping which preserves the two algebraic operations of an *als* that is, $T : X \to Y$ is a bijective linear operator. Y is then called *isomorphic* with X.

LEMMA 3.1. *Let T be a linear operator from an als X into an als Y.*

(1) $T(V_X) \subset V_Y$, $T(W_X) \subset W_Y$.
(2) *If $X = V_X + W_X$, then $T(X) = T(V_X) + T(W_X)$. In particular, if T is an isomorphism, then $Y = T(X) = V_Y + W_Y$.*
(3) *If T is an isomorphism and X has a basis B, then $T(B)$ is a basis for Y.*

PROOF. (1) Suppose $x \in V_X$. Then $x - x = 0$. Therefore,

$$T(x) - T(x) = T(x - x) = T(0) = 0.$$

which shows that $T(x) \in V_Y$. Similarly, suppose $x \in W_X$. Then $-x = x$. Therefore,

$$-T(x) = T(-x) = T(x),$$

which shows that $T(x) \in W_Y$.

(2) If $X = V_X + W_X$, then every element $x \in X$ is of the form $v + w \in V_X + W_X$, and $T(x) = T(v) + T(w)$.

(3) Let $y \in Y$, and let $T(x) = y$. Then x has a positive representation with respect to the basis B, say $x = \sum \lambda_j x_j$, where $x_j \in B$, $\lambda_j > 0$ for those j for which $x_j \in B \setminus V_X$. Since T is a linear operator, $y = T(x) = T(\sum \lambda_j x_j) = \sum \lambda_j T(x_j)$. So, $y = \sum \lambda_j T(x_j)$ is a positive representation of y with respect to the subset $T(B)$. It remains to show that such a representation is unique. But, if $y = \sum \mu_j T(x_j)$ is another positive representation, then $x' = \sum \mu_j x_j$ is another preimage of y, because $T(x') = T(\sum \mu_j x_j) = \sum \mu_j T(x_j) = y$. Since T is one-to-one, we must have $x = x'$, and hence $\lambda_j = \mu_j$ for all j. This establishes the uniqueness of the representation of y.

Let X be an als. A functional $f : X \to \mathbb{R}$ is called an ***almost linear functional*** if the conditions (3.1) – (3.3) are satisfied.

$$f(x+y) = f(x) + f(y) \quad (x, y \in X) \tag{3.1}$$

$$f(\lambda x) = \lambda \cdot f(x) \quad (\lambda \geq 0,\ x \in X) \tag{3.2}$$

$$f(w) \geq 0 \quad (w \in W_X). \tag{3.3}$$

The functional $f : X \to \mathbb{R}$ is called a *linear functional* on X if it satisfies (3.1), and (3.2) for each $\lambda \in \mathbb{R}$. Then (3.3) is also satisfied. Note that an almost linear functional is not a linear operator from X to $\mathbb{R}$, but a linear functional is a linear operator.

Let $X^{\#}$ be the set of all almost linear functionals defined on the *als* X. We define two operations $s : X^{\#} \times X^{\#} \to X^{\#}$ and $m : \mathbb{R} \times X^{\#} \to X^{\#}$ as follows:

$$s(f_1, f_2)(x) = f_1(x) + f_2(x) \qquad (f_1, f_2 \in X^{\#}),$$

$$m(\lambda, f)(x) = f(\lambda x) \qquad (\lambda \in \mathbb{R},\ f \in X^{\#})$$

for all $x \in X$. Clearly, $s(f_1, f_2) \in X^{\#}$, $m(\lambda, f) \in X^{\#}$, and s, m satisfy $(L_1) - (L_8)$ with $0 \in X^{\#}$ being the functional which is 0 at each $x \in X$. Therefore $X^{\#}$ is an *als*. $X^{\#}$ is called the ***algebraic dual space*** of the *als* X. We denote $s(f_1, f_2)$ by $f_1 + f_2$ and $m(\lambda, f)$ by $\lambda \circ f$.

PROPOSITION 3.2. *Let X be an almost linear space. Then*

$$X^{\#} = W_{X^{\#}} + V_{X^{\#}}.$$

PROOF. For any $f \in X^{\#}$, define two functionals f_1, f_2 on X by

$$f_1 = f - (-1 \circ f), \quad f_2 = f + (-1 \circ f).$$

Then $f_1 \in V_{X^{\#}}$ since $f_1 + (-1 \circ f_1) = 0$, and $f_2 \in W_{X^{\#}}$ since $-1 \circ f_2 = f_2$. Clearly $f = \frac{1}{2} \circ f_1 + \frac{1}{2} \circ f_2$. Therefore $X^{\#} = W_{X^{\#}} + V_{X^{\#}}$.

Let B be a symmetric basis for an *als* X. For each $x_i \in B$, define

$$x_i' : X \to \mathbb{R} \tag{3.4}$$

as follows: For a positive representation $x = \sum_{j=1}^{n} \lambda_j x_j \in X$, let $x_i'(x) = \lambda_i$.

THEOREM 3.3. *Let B be a symmetric basis for an als X.*

(1) *The map x_i' defined by (3.4) is an almost linear functional on X.*

(2) *The map $T : X \to X^\#$ defined by*

$$T\left(\sum_{j=1}^{n} \lambda_j x_j\right) = \sum_{j=1}^{n} \lambda_j \circ x_j'$$

is an injective linear operator. In particular, it maps V_X, W_X into $V_{X^\#}$, $W_{X^\#}$, respectively.

PROOF. (1) Clearly, $x_i'(x+y) = x_i'(x) + x_i'(y)$ and $x_i'(\lambda x) = \lambda x_i'(x)$ for $x, y \in X$, $\lambda \geq 0$.

We need to show that $x_i'(w) \geq 0$ for $w \in W_X$. Since B is a symmetric basis, $\{x - x : x \in B \setminus V_X\}$ is a basis for W_X by Theorem 2.2 (2). Therefore,

$$w = \sum_{j \in J} \lambda_j (x_j - x_j) = \sum_{j \in J} (\lambda_j x_j + \lambda_j(-x_j))$$

with all $x_j \in B \setminus V_X$ and $\lambda_j > 0$. Clearly, $x_i'(w) = 0$ if $i \notin J$. Otherwise, $x_i = x_k$ or $x_i = -x_k$ for some $k \in J$. Either case, we have $x_i'(w) = \lambda_k$ or $2\lambda_k$.

(2) We only need to show that T is injective. Let $\sum_i \lambda_i x_i$ be a positive representation. Then

$$\begin{aligned} T\Big(\sum_i \lambda_i x_i\Big)(x_j) &= (\sum_i \lambda_i \circ x_i')(x_j) \\ &= \sum_i x_i'(\lambda_i x_j) \\ &= \sum_i \lambda_i \delta_{ij} \\ &= \lambda_j. \end{aligned}$$

Therefore, $T(\sum_i \lambda_i x_i)(x_j) = T(\sum_i \mu_i x_i)(x_j)$ for all j implies $\lambda_j = \mu_j$ for all j. Consequently, $\sum_i \lambda_i x_i = \sum_i \mu_i x_i$.

An almost linear subspace Γ of $X^\#$ is said to be *total* over X if the relations $x_1, x_2 \in X$, $f(x_1) = f(x_2)$ for each $f \in \Gamma$ imply that $x_1 = x_2$.

Note that $X^\#$ may be not total over the *als* X [Examples 2.1(1)].

THEOREM 3.4. *If an als X has a basis, then $X^{\#}$ is total over X.*

PROOF. Let B be a symmetric basis for X and let $x, y \in X$ such that $f(x) = f(y)$ for each $f \in X^{\#}$. By definition of basis we have $x = \sum_{i=1}^{n} \lambda_i x_i$, $y = \sum_{i=1}^{n} \mu_i x_i$ where $x_i \in B$ and $\lambda_i, \mu_i \geq 0$ if $x_i \notin V_X$. For each $x_i \in B$, let x_i' be the almost linear functional on X defined by (3.4). Then $x_i'(x) = \lambda_i$ and $x_i'(y) = \mu_i$, $i = 1, 2, ..., n$. Since $f(x) = f(y)$ for all $f \in X^{\#}$, $\lambda_i = \mu_i$, $i = 1, 2, ..., n$. Hence $x = y$.

We may go a step further and consider the algebraic dual $(X^{\#})^{\#}$ of $X^{\#}$, whose elements are the almost linear functionals defined on $X^{\#}$. We denote $(X^{\#})^{\#}$ by $X^{\#\#}$ and call it the *second algebraic dual space* of the *als* X.

For $x \in X$ let Q_x be the functional on $X^{\#}$ defined, as in the case of a *ls*, by

$$Q_x(f) = f(x) \quad (f \in X^{\#}). \tag{3.5}$$

Then Q_x is an almost linear functional on $X^{\#}$. Hence Q_x is an element of $X^{\#\#}$, by definition of $X^{\#\#}$. This defines a mapping

$$C : X \to X^{\#\#} \tag{3.6}$$

by $C(x) = Q_x$. C is called the *canonical mapping* of X into $X^{\#\#}$. Of course, C is defined for every *als* X, even when it does not have a basis.

Clearly, the canonical mapping C of X into $X^{\#\#}$ defined by (3.6) is a linear operator.

If X is isomorphic with an almost linear subspace of an *als* Y, we say that X is *embeddable* in Y. In contrast to the case of a *ls*, *als* X need not be embeddable in $X^{\#\#}$. Indeed, in Examples 2.1(1) $X^{\#} = \{0\}$. Hence $X^{\#\#} = \{0\}$. However, in the case when the *als* X has a basis, X is embeddable in $X^{\#\#}$ by the following theorem.

THEOREM 3.5. *If an als X has a basis, then the canonical mapping $C : X \to X^{\#\#}$ is injective.*

PROOF. If an *als* X has a basis, then $X^{\#}$ is total over X by Theorem 3.4. Hence, we have $Q_x = Q_y \iff Q_x(f) = Q_y(f)$ for all $f \in X^{\#} \iff f(x) = f(y)$ for all $f \in X^{\#} \iff x = y$. Therefore C is injective.

If the canonical mapping C of an *als* X into $X^{\#\#}$ defined by (3.6) is an isomorphism, then X is said to be *algebraically reflexive.* From Lemma 3.1 and Proposition 3.2, we have

THEOREM 3.6. *If an als X is algebraically reflexive, then*

$$X = W_X + V_X.$$

LEMMA 3.7. *Let $X = W_X + V_X$ be an als which has a finite basis. If $B = \{x_1, ..., x_n\}$ is a basis for the als X such that $B \subset W_X \cup V_X$, then $B' = \{x'_1, ..., x'_n\}$ given by (3.4) is a basis for the algebraic dual $X^{\#}$ of X.*

PROOF. In the light of Lemma 3.1 (3), it is enough to prove that the linear operator $T : X \to X^{\#}$ is surjective. For a given $f \in X^{\#}$, let $f(x_i) = \alpha_i$ for each $x_i \in B$. If $x_i \notin V_X$ then $x_i \in W_X$, so that $\alpha_i \geq 0$. (This step is not true without the assumption $B \subset W_X \cup V_X$ which is guaranteed by $X = W_X + V_X$). For every positive representation $x = \sum_{i=1}^{n} \lambda_i x_i \in X$ (so that $\lambda_i \geq 0$ if $x_i \notin V_X$), we have

$$f(x) = \sum_{i=1}^{n} \lambda_i \alpha_i$$

since f is an almost linear functional on X. Let $x_i \in V_X$. Then $x'_i \in V_{X^{\#}}$ by Theorem 3.3(2). So, x'_i is a linear functional on X.

Now we shall calculate $\sum_{i=1}^{n} (\alpha_i \circ x'_i)(x)$. Suppose $x_i \in V_X$. Then $x'_i(\alpha_i x) = \alpha_i \cdot x'_i(x)$ since x'_i is a linear functional. Suppose $x_i \in W_X$. Then $x'_i(\alpha_i x) = \alpha_i \cdot x'_i(x)$ since $\alpha_i > 0$. In any case, we have $x'_i(\alpha_i x) = \alpha_i \cdot x'_i(x)$. Consequently,

$$\sum_{i=1}^{n} (\alpha_i \circ x'_i)(x) = \sum_{i=1}^{n} x'_i(\alpha_i x) = \sum_{i=1}^{n} \alpha_i \cdot x'_i(x) = \sum_{i=1}^{n} \alpha_i \lambda_i.$$

We have shown that

$$f = \sum_{i=1}^{n} \alpha_i \circ x'_i = T\left(\sum_{i=1}^{n} \alpha_i x_i\right).$$

Therefore, T is surjective, and B' is a basis for the *als* $X^{\#}$.

REMARK. In Lemma 3.7, $X = W_X + V_X$ is essential. Indeed, in Examples 2.1(2) $Y = \{[a,b] : a \leq 0,\ b \geq 0\}$ is an *als* and $B = \{b_1 = [-1,0], b_2 = [0,1]\}$ is a basis for Y. Note that $W_Y = \{[-a,a] : a \geq 0\}$, $V_Y = \{\{0\}\}$ and $Y \neq W_Y + V_Y$. But $B' = \{b'_1, b'_2\}$ is not a basis for $Y^{\#}$. For example, the element $f = b'_1 - (-1 \circ b'_1) \in Y^{\#}$ cannot be written as a positive representation of b'_1 and b'_2: Suppose $B' = \{b'_1, b'_2\}$ were a basis for $Y^{\#}$. Then $f = \alpha_1 \circ b'_1 + \alpha_2 \circ b'_2$ with both α_i's non-negative. Now $(\alpha_1 \circ b'_1 + \alpha_2 \circ b'_2)(b_2) = \alpha_2 \geq 0$. However, $f(b_2) = -1$. Therefore, such α_i's cannot exist.

THEOREM 3.8. *Let an als X have a finite basis and $X = W_X + V_X$. Then X is algebraically reflexive.*

PROOF. We may assume that $B = \{x_1, x_2, \ldots, x_n\}$ is a basis for the *als* X such that $B \subset W_X \cup V_X$ by Theorem 2.2(5). Let $T : X \to X^{\#}$ be the linear operator given in Theorem 3.3. Then the set $B' = \{x'_1, x'_2, \ldots, x'_n\}$, $T(x_i) = x'_i$ is a basis for the *als* $X^{\#}$ by Lemma 3.7. We apply the same theorem to the *als* $X^{\#}$ to get a linear operator. Let $T' : X^{\#} \to X^{\#\#}$ be the linear operator. Then the set $B'' = \{x''_1, x''_2, \ldots, x''_n\}$, $T'(x'_i) = x''_i$ is a basis for the als $X^{\#\#}$. Clearly, $Q_{x_i} = T'T(x_i)$. Hence the canonical mapping C defined by (3.6) is an isomorphism, since C is a composite of two isomorphisms.

By Theorem 3.6 and Theorem 3.8, we have

COROLLARY 3.9. *Let an als X have a finite basis. Then $X = W_X + V_X$ if and only if X is algebraically reflexive.*

4. Reflexivity of NALS

A *norm* on the *als* X is a functional $\|\cdot\| : X \to \mathbb{R}$ satisfying the conditions $(N_1)-(N_3)$ below. Let $x, y, z \in X$ and $\lambda \in \mathbb{R}$. (N_1) $\|x-z\| \leq \|x-y\| + \|y-z\|$; (N_2) $\|\lambda x\| = |\lambda|\|x\|$; (N_3) $\|x\| = 0$ iff $x = 0$.

Using (N_1) we get

$$(4.1) \qquad \|x+y\| \leq \|x\| + \|y\| \quad (x, y \in X)$$

$$(4.2) \qquad \|x-y\| \geq |\|x\| - \|y\|| \quad (x, y \in X)$$

By the above axioms it follows that $\|x\| \geq 0$ for each $x \in X$.

An almost linear space X together with $\|\cdot\| : X \to \mathbb{R}$ satisfying $(N_1) - (N_3)$ is called a *normed almost linear space* (*nals*).

When X is a *nals*, for $f \in X^{\#}$ define, as in the case of a normed linear space,

$$(4.3) \qquad \|f\| = \sup\{|f(x)| : x \in X, \|x\| \leq 1\},$$

and let

$$X^* = \{f \in X^{\#} : \|f\| < \infty\}.$$

Then X^* is a normed almost linear space[3], called the *dual space* of X. We denote the dual space $(X^*)^*$ of X^* by X^{**} and call it the *second dual space* of X.

PROPOSITION 4.1[4]. *Let* $(X, \|\cdot\|)$ *be a nals. Then for each* $x \in X$ *there exists* $f_x \in X^*$ *with* $\|f_x\| = 1$ *such that* $f_x(x) = \|x\|$.

PROPOSITION 4.2[2]. *Let a nals* X *have a finite basis. Then*

$$X^{\#} = X^*.$$

An *isomorphism* T of a *nals* X onto a *nals* Y is a bijective linear operator $T : X \to Y$ which preserves the norm, that is, for all $x \in X$,

$$\|T(x)\| = \|x\|.$$

X is then called *isomorphic* with Y. If a *nals* X is isomorphic with an almost linear subspace of a *nals* Y, then we say that X is *embeddable* in Y.

For $x \in X$ let Q_x be the functional on X^* defined, as in the case of an *als*, by

$$Q_x(f) = f(x) \quad (f \in X^*). \tag{4.4}$$

Then Q_x is an almost linear functional on X^* and

$$\|Q_x\| \leq \|x\|. \tag{4.5}$$

Hence Q_x is an element of X^{**}, by definition of X^{**}. This defines a mapping

$$C : X \rightarrow X^{**} \tag{4.6}$$

by $C(x) = Q_x$. C is called the *canonical mapping* of X into X^{**}.

If the canonical mapping C of a *nals* X into X^{**} defined by (4.6) is an isomorphism, then X is said to be *reflexive*.

As in the case of an *als*, $X^* = W_{X^*} + V_{X^*}$ for the *nals* X. Thus we have the following theorem.

THEOREM 4.3. *If a nals X is reflexive, then $X = W_X + V_X$.*

THEOREM 4.4. *Let a nals X have a finite basis and $X = W_X + V_X$. Then X is reflexive.*

PROOF. If a *nals* X has a finite basis, then $X^\# = X^*$ by Proposition 4.2. Hence the canonical mapping C defined by (4.6) is bijective linear operator by Theorem 3.8. We must show that C preserves the norm, that is, $\|Q_x\| = \|x\|$ where Q_x is the functional on X^* defined by (4.4). From (4.5) $\|Q_x\| \leq \|x\|$. And, Proposition 4.1 implies that $\|Q_x\| = \sup\{|Q_x(f)| : f \in X^*, \|f\| \leq 1\} = \sup\{|f(x)| : f \in X^*, \|f\| \leq 1\} \geq f_x(x) = \|x\|$. Hence $\|C(x)\| = \|x\|$. This completes the proof.

THEOREM 4.5. *Let a nals X have a finite basis. Then $X = W_X + V_X$ if and only if X is reflexive.*

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References

1. Dunford, N. and Schwarz, J., *Linear operators. Part I, Pure and applied Mathematics, 7.*, New York, London, Interscience, 1958.
2. Godini, G., *An approach to generalizing Banach spaces: Normed almost linear spaces*, Proceedings of the 12th Winter School on Abstract Analysis (Srni 1984). Suppl. Rend. Circ. Mat. Palermo II. Ser. 5 (1984), 33-50.
3. ______, *A framework for best simultaneous approximation: Normed almost linear spaces*, J. Approximation Theory **43** (1985), 338-358.
4. ______, *On Normed Almost Linear Spaces*, Math. Ann. **279** (1988), 449-455.
5. Milman, P. D., *On best simultaneous approximation in normed linear spaces*, J. Approx. Theory **20** (1977), 223-238.

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