Comm. Korean Math. Soc. **10** (1995), No. 4, pp. 839–847

SEMISIMPLE ARTINIAN LOCALIZATIONS RELATED WITH V-RINGS

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ABSTRACT. For the given torsion theory τ, we study some equivalent conditions when the localized ring R_τ be semisimple artinian (Theorem 4). Using this, if R_τ is semisimple artinian ring, we study when does the given ring R become left V-ring?

1. Preliminaries

Throughout the following R will denote an associative ring with non-zero unit element, and R-Mod will denote the category of all left R-modules.

Notation and terminology concerning (hereditary) torsion theories on R-Mod will follow ([7]). In particular, if τ is a torsion theory on R-Mod, for a given left R-module M, we denote by $\tau(M)$ the unique largest submodule of M which is τ-torsion. If $E(M)$ is the injective hull of a left R-module M then we define the submodule $E_\tau(M)$ of $E(M)$ by $E_\tau(M)/M = \tau(E(M)/M)$. The module of quotients of M with respect to τ, denoted by $Q_\tau(M)$, is then defined to be $E_\tau(M/\tau(M))$. Note that, in particular, if M is τ-torsionfree then $Q_\tau(M) = E_\tau(M)$, and this is a left R-module containing M as a large submodule. In general, we have a canonical R-homomorphism from M to $Q_\tau(M)$ obtained by composing the canonical surjection from M to $M/\tau(M)$ with the inclusion map into $Q_\tau(M)$.

If R_τ is the endomorphism ring of the left R-module $Q_\tau({}_RR)$ then $Q_\tau(M)$ is canonically a left R_τ-module for every left R-module M and

Received August 1, 1995. Revised September 28, 1995.

1991 AMS Subject Classification: 16D60 (16D50, 16N).

Key words: semisimple artinian localizations V-ring, semicocritical module, linked, Goldie ring, semimaximal prime torsion Theory.

This paper was supported in part by NON DIRECTED RESEARCH FUND, Korea Research Foundation, 1993.

the canonical map $R \longrightarrow R_\tau$ is a ring homorphism, the ring R_τ is called the ring of quotients or localization of R at τ. A torsion theory on R-Mod is said to be faithful if and only if R, considered as a left module over itself, is τ-torsionfree. In this case, R is canonically subring of R_τ.

A submodule N of M is called τ-closed (τ-dense) in M if M/N is τ-torsionfree (τ-torsion). A module M is called τ-cocritical if it is τ-torsion free, but every proper homomorphic image of it is τ-torsion. A module M will be called τ-semicocritical if there exists a finite set $K_1, K_2, \cdots, K_n$ of submodules of M such that $\cap_{i=1}^{n} K_i = 0$ and M/K_i is τ-cocritical for each $i = 1, 2, \cdots, n$. This concept is closely related to the idea of an α-critical module in the study of Krull dimension ([2]). If M is cocritical (resp. semicocritical) with respect to the torsion theory cogenerated by the injective hull of M, it is simply called cocritical (resp. semicocritical). Any module that is τ-cocritical (resp. τ-semicocritical) for some τ is necessarily a cocritical (resp. semicocritical). (cf. [10] or [11])

A module M is τ-artinian if it has descending chain conditions on τ-closed submodules. R is called τ-artinian (resp. (τ-) cocritical, (τ-semicocritical) if it is τ-artinian (resp. (τ-) cocritical, (τ-) semicocritical) as an R-module.

If X is a subset of an R-module, then $ann_R(X)$ will denote the elements of R that annihilate the set X. (If there is no chance for confusion, the R will not be written.) A module M is called finitely annihilated if $ann(M)$ equals the annihilator of a finite subset of M. Observe that M is finitely annihilated if and only if there exists an embedding of $R/ann(M)$ into a finite direct sum of copies of M. A module M is called Δ-module if M has the DCC on annihilators of subsets of M.

We will use the following result several times.

LEMMA 1. ([11], Proposition 4.5) *Let M be τ-torsionfree and τ-artinian. Then the following statements are equivalent.*

(1) $R/annM$ is left τ-artinian.
(2) M is finitely annihilated.
(3) M is Δ-module.

Finally, for any module M, $\chi(M)$ will denote the torsion theory cogenerated by $E(M)$; i.e., the largest torsion theory for which M is torsionfree.

Recall that a torsion theory τ is called perfect if all the R_τ modules are τ-torsion free. For such torsion theories we have $Q_\tau(M) \simeq Q_\tau(R) \otimes_R M$ for every $M \in R$-Mod. Many characterizations of perfect torsion theories are known (cf. [7] Proposition 45.1)

One result we will need is that if R_τ is a semisimple ring, then τ is a perfect torsion theory [[8], Proposition 2.3].

2. Semisimple Artinian Localizations

We characterize those torsion theories τ for which the ring of quotients R_τ is a semisimple ring (i.e. a direct sum of rings each of which is a finite matrix ring over a division ring). Teply and Shapiro characterized R_τ to be semisimple in [9], we have more characterizations involves the existence of certain semicocritical modules.

An ideal of R is called τ-primitive if it is the annihilator of a τ-cocritical module, while the intersection of such ideals is called a τ-semiprimitive ideal. We note that the annihilator of any nonzero τ-semicocritical module is τ-semiprimitive ideal (cf. [11], Lemma 2.1).

We say the ring R is τ-primitive (τ-semiprimitive) if the zero ideal is τ-primitive (respectively τ-semiprimitive). If R is τ-primitive (resp. τ-semiprimitive), then it embeds in a (possibly infinite) power of a τ-cocritical (resp. τ-semicocritical) module. Hence R must be τ-torsionfree.

For information on τ-primite (or τ-semiprimitive) ideals the reader may refer to [7] or [12].

In order to study when R_τ is semisimple artinian ring, as indicated in [9], one has to wory about the existence of homomorphisms between modules of the form $E(M)$ snd $E(N)$, where M and N are nonisomorphic simple modules. This is something that can not happen when R is a commutative artinian ring.

Given a torsion theory τ such that R_τ is simple artinian, there exists a unique (up to isomorphism) indecomposable, injective τ-torsionfree module E. We will call this the associated injective of τ. Given two such torsion theories τ_1 and τ_2 we will say they are linked if, either $\mathrm{Hom}(E_1, E_2) \neq 0$ or $\mathrm{Hom}(E_2, E_1) \neq 0$, where E_i is the associated injective of τ_i. If they are not linked, we will say they are unlinked. We can findout in [9] or [11] for a discussion of such links.

If τ_1 and τ_2 are two torsion theories, then their intersection, denoted $\tau_1 \wedge \tau_2$, is the torsion theory in which a module is torsion if and only if it is torsion with respect to both τ_1 and τ_2. We also use the following two Lemmas frequently.

LEMMA 2. ([1] and [10]) *If R is τ-artinian. then we have the following;*

(1) There are only finitely many non-isomorphic indecomposable injective τ-torsionfree modules $E_1, E_2, \cdots, E_n$.

(2) $\tau = \chi({}_RE(R/\tau(R)))$.

(3) Let $\tau_i = \chi(E_i)$, where each E_i appears in (1) then each τ_i is unlinked.

PROOF. (1) [Teply, [11] Remark 3.4]

(2) [Benander, [2] Theorem 4.3(4)]

(3) If there is a map $f : E_i \longrightarrow E_j, i \neq j$ clearly this is not monic. But then $E_i/kerf$ is τ-torsion, as E_i is τ-cocritical so $f = 0$.

LEMMA 3. ([11] Proposition 1.1) *A module M is τ-semicocritical if and only if the following condition hold:*

(1) M is τ-artinian

(2) M is τ-torsionfree

(3) M/N is τ-torsion, if N is essential in M.

THEOREM 4. *Let τ be a torsion theory on R-Mod, then the following statements are equivalent.*

(1) R_τ is a semisimple artinian ring.

(2) R is τ-atinian and $\tau(R)$ is τ-semiprimitive.

(3) $\tau = \tau_1 \wedge \cdots \wedge \tau_n$, where each R_{τ_i} is simple artinian and the set $\{\tau_1, \cdots, \tau_n\}$ is pairwise unlinked.

(4) There exists a semicocritical, finitely annihilated, injective R-module E with $\chi(E) = \tau$.

(5) There exists a semicocritical module M such that $\chi(M) = \tau$, M is finitely annihilated and $ann(M) = ann(E(M))$.

PROOF. The equivalences of (1), (2) and (3) are in [[9], Theorem 1.2]. (1) $\longrightarrow$ (4). Since R_τ is semisimple artinian ring, τ is a perfect torsion theory; so τ is cogenerated by the R_τ-modules that are the injective hulls of semisimple artinian R_τ-modules (when considered as left R-modules). Since R_τ is artinian and τ is perfect torsion theory, R is τ-artinian. By Lemma 2 (1), there are only finitely many non-isomorphic indecomposable injective τ-torsionfree modules, which we call $E_1, E_2, \cdots, E_n$. Note that each E_i is an injective, simple R_τ-module. Let $E = E_1 \oplus E_2 \oplus \cdots \oplus E_n$, so if $\tau_i = \chi(E_i)$ we see that $\tau = \chi(E) = \tau_1 \wedge \tau_2 \wedge \cdots \wedge \tau_n$. And each E_i is an injective τ-cocritical R-module, thus E is a τ-semicocritical R-module.

Note that E is τ-torsionfree and τ-artinian and $R/ann(E)$ is τ-artinian also (since $R/ann(E)$ is a homomorphic image of τ-artinian ring R). Now apply Lemma 1, we have that E is finitely annihilated.

(4) $\longrightarrow$ (5). This is immediate. (5) $\longrightarrow$ (2) Let M be a semicocritical module,and let $I = ann_R(M)$ which is $\tau = \chi(M)$-semiprimitive by [14, Lemma 4.1.]. Since M is finitely annihilated, R/I can be embedded in M^n for some integer n. Thus R/I is τ-torsionfree (since M is τ-semicocritical). So $\tau(R) \subseteq I$. Now let $f : I \longrightarrow E(M)$. Extend f to a map $g : R \longrightarrow E(M)$. But $g(I) = Ig(1) = 0$ since $ann(M) = ann(E(M))$. Thus I is τ-torsion. So $\tau(R) = I$ which implies that $\tau(R)$ is τ-semiprimitive. Since M^n is τ-artinian, so is R.

3. Semisimple Artinian Localizations and V-rings

Recall that a ring R is a V-ring if all the simple R-modules are injective. For commutative rings the following conditions are well known to be equivalent: i) R_p is a field for every maximal ideal P; ii) R is fully idempotent (i.e., every ideal is idempotent); iii) R is a von Neumann regular ring; iv) R is a V-ring. It is equally well-known that for noncommutative rings, iii) or iv) imply ii), but that there are no other implications. For a general discussion of the relationship between the last three condition can be found in [6]. Teply and Shapiro proved that i) and ii) imply iv), also i) and iv) imply that every ideal of R is the intersection of maximal ideals under the condition that R_τ is simple artinian ring and certain torsion theory τ. Still we are able to get more general results related with V-rings under the condition that R_τ is semisimple artinian and suitable torsion theory τ.

Follow the idea of [8], we will call τ a maximal prime (resp. semimaximal prime) torsion theory if $\tau = \chi(M)$, where M is simple (resp.

semisimple artinian) module. Note that maximal prime torsion theories on R-Mod are semimaximal prime.

LEMMA 5. *Let τ be a torsion theory and R_τ be semisimple artinian ring, then $R/\tau(R)$ is non-singular ring.*

PROOF. Since τ is perfect torsion theory and R_τ is artinian, we see that homomorphic image $\bar{R} = R/\tau(R)$ is τ-artinian. We know that τ induces a torsion theory $\bar{\tau}$ on $R/\tau(R)$-Mod as follows; $_{\bar{R}}M$ is $\bar{\tau}$-torsion if $_RM$ is τ-torsion. We can prove that $R/\tau(R)$ is $\bar{\tau}$-semicocritical, thus $\bar{\tau}$ is a generalization of Goldie torsion theory. Thus $\bar{\tau}$-torsionfree left $\bar{R}$-module $R/\tau(R)$ is Goldie torsionfree i.e., $R/\tau(R)$ is non-singular ring.

PROPOSITION 6. *Assume that R_τ is a semisimple artinian ring for every semimaximal prime torsion theory τ. Futhermore assume that each pair of semimaximal torsion theories is unlinked. Then $R/\tau(R)$ is a finite product of simple left V-rings.*

PROOF. Since τ is perfect torsion theory and R_τ is artinian, we see that $\bar{R}$ is τ-artinian. We know that τ induces a torsion theory $\bar{\tau}$ on $\bar{R} = R/\tau(R)$ as in the Lemma 5. From the fact that $Q_\tau(M)$ is a module over R_τ and hence a module over $\bar{R} = R/\tau(R)$. Thus we can write $\bar{R}_{\bar{\tau}} = \bar{R}_\tau = R_\tau$.

Now by Lemma 2, $\tau = \chi(_RE(R))$. Letting $\gamma = \chi(E(_{\bar{R}}\bar{R}))$, we have that $\bar{\tau} = \chi(_{\bar{R}}\bar{R}) = \gamma$. Since $\bar{R}_{\bar{\tau}} = R_\tau$. Thus R_τ is the maximal ring of quotients of $\bar{R}$, which is non-singular ring. So we have that $\bar{R}$ is an order in a semisimple artinian ring.

Now we want to show that $\bar{R}$ is left V-ring. Let E be the injective hull of a simple module S and let $\pi = \chi(E)$. π is a semimaximal prime torsion theory, so $\bar{R}_\pi$ is semisimple artinian ring. By Theorem 4, E must be a semicocritical module, but in here E is cocritical module. Let H be a nonzero submodule of E. Since the semimaximal prime torsion theories are unlinked, then $\mathrm{Hom}(E/H, E') = 0$ for any injective hull of a semisimple module. This can only happen if $H = E$. Thus E is a simple module, which proves that $\bar{R}$ is a left V-ring. Now by ([5], Theorem 7.36 A), $\bar{R}$ is a finite product of simple left V-rings.

COROLLARY 7. *Under the same hypothesis of Proposition 6, $R/\tau(R)$ is finite direct sum of simple Goldie left V-rings.*

PROOF. Since $R/\tau(R)$ is Goldie ring, now apply ([3], 5.16), we have the result.

COROLLARY 8. *Assume that R_τ is a simple artinian ring for every maximal prime torsion theory τ. Furthermore assume that each pair of maximal torsion theory is unlinked. Then $R/\tau(R)$ is a simple Goldie left V-ring.*

PROPOSITION 9. *Assume that R_τ is semisimple artinian for every semimaximal prime torsion theory τ. Then every factor ring of R also has this property.*

PROOF. Let $\bar{R} = R/I$ be an arbitrary factor ring of R and let M be a semisimple artinian $\bar{R}$-module. Then M is also semisimple artinian R-module. If $\tau = \chi({}_RM)$, then the torsion theory $\bar{\tau}$ on $\bar{R}$-Mod, which we denote $\bar{\tau}$ is $\chi({}_{\bar{R}}M)$. If $E = E_R(M)$, then we know that $E_{\bar{R}}(M) = \{x \in E | Ix = 0\}$. Since ${}_RE$ is semicocritical module. Furthermore $\bar{R}$ is τ-artinian, as it is a homomorphic image of R. Thus $\bar{R}$ is $\bar{\tau}$-artinian. By Lemma 1, $E_{\bar{R}}(M)$ finitely annihilated as left $\bar{R}$-module. Therefore by Theorem 4, $\bar{R}_{\bar{\tau}}$ is semisimple artinian.

PROPOSITION 10. *Assume that R_τ is semisimple artinian for every semimaximal prime torsion theory τ. If I is a semiprimitive ideal of R, then R/I is left artinian (so I must be the intersection of maximal ideals)*

PROOF. By Proposition 9, we can replace R/I with R and assume that R is a semiprimitive ring. Let M be a faithful semisimple R-module and let $\tau = \chi(M)$.

Then R is τ-torsionfree, since R embeds in a product of copies of M. Since R_τ is artinian and τ is perfect torsion theory, R is τ-artinian. Thus $R/ann_R(M)$ is τ-artinian as an homomorphic image of R. By Lemma 1, ${}_RM$ is finitely annihilated; i.e., R embeds in M^n for some positive integer n. Since M is semisimple artinian, so is M^n. We have the result.

Proposition 11. *Let R be a left V-ring such that R_τ is a semisimple artinian ring for every semimaximal prime torsion theory τ. Then every ideal of R is the intersection of maximal ideals of R.*

Proof. Let M be a semisimple artinian module, and $\tau = \chi(M)$. Since R is a left V-ring, M is also injective as finite direct sum of injective modules. Clearly $\tau(R)$ is τ-semiprimitive and must equal to the annihilator of M (or $E(M)$) (in the proof (4) $\longrightarrow$ (5) in Theorem 4). And $R/\tau(R)$ embeds in M^n for some finite integer n. Thus $\tau(R)$ is the intersection of maximal ideals by Proposition 9. Use the unlinkedness of semimaximal prime torsion theories, we have that $\cap\tau(R) = 0$ as τ through all the semimaximal prime torsion theories. So O is a semimaximal ideal. Consequently, by Proposition 10 and the fact that factor rings of left V-rings are again left V-rings, every ideal of R is semimaximal.

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