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WEAKLY POSITIVE IMPLICATIVE BCI-ALGEBRAS

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ABSTRACT. We introduce the concept of weakly positive implicative ideals in BCI-algebras and give some characterizations of weakly positive implicative BCI-algebras and weakly positive implicative ideals.

In 1978, K. Iséki and S. Tanaka [3] introduced the concept of positive implicative BCK-algebras. M. A. Chaudhry [1] defined the notion of weakly positive implicative BCI-algebras as a generalization of positive implicative BCK-algebras and gave a characterization of a weakly positive implicative BCI-algebra. In this note, we establish the notion of weakly positive implicative ideals in BCI-algebras, and investigate some properties of it. Some characterizations of weakly positive implicative BCI-algebras and weakly positive implicative ideals are given. First of all we recall some definitions and results concerning BCI-algebras.

An algebra $(X;*,0)$ of type $(2,0)$ is called a *BCI-algebra* if it satisfies the following axioms:

(I) $((x*y)*(x*z))*(z*y)=0$,

(II) $(x*(x*y))*y=0$,

(III) $x*x=0$,

(IV) $x*y=0$ and $y*x=0$ imply $x=y$.

A partial ordering $\leq$ on a BCI-algebra X can be defined by $x \leq y$ if and only if $x*y=0$.

If a BCI-algebra X satisfies the identity

(V) $0*x=0$ for all $x \in X$,

then X is called a *BCK-algebra*.

In a BCI-algebra X, the following hold:

(1) $x \leq 0$ implies $x=0$.

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(2) $x * 0 = x$.
(3) $(x * y) * z = (x * z) * y$.
(4) $0 * (x * y) = (0 * x) * (0 * y)$.
(5) $x * (x * (x * y)) = x * y$.
(6) $((x * z) * (y * z)) * (x * y) = 0$.
(7) $x * y = 0$ implies $(x * z) * (y * z) = 0$ and $(z * y) * (z * x) = 0$.

A non-empty subset I of a BCI-algebra X is called an *ideal* of X if it satisfies:

(i) $0 \in I$,
(ii) $x * y \in I$ and $y \in I$ imply $x \in I$.

An ideal I of a BCI-algebra X is called *closed* if $x \in I$ implies $0 * x \in I$.

Any ideal I of a BCI-algebra X has the following property:

(8) $x \in I$ and $y \leq x$ imply $y \in I$.

C. S. Hoo [2] gave a characterization of closed ideal as follows: An ideal I of a BCI-algebra X is closed if and only if I is a subalgebra of X.

J. Meng and X. L. Xin [5] introduced the notion of atoms. An element a of a BCI-algebra X is called an *atom* if $x * a = 0$ implies $x = a$ for all $x \in X$. Let $L(X)$ denote the set of all atoms of X. For all a in $L(X)$, $V(a) = \{x \in X : a * x = 0\}$ is called a *branch* of X.

PROPOSITION 1 ([5]). *Let X be a BCI-algebra. Then the following hold:*

(i) *For any $x \in X$, $0 * (0 * x) \in L(X)$ and $x \in V(0 * (0 * x))$.*
(ii) *If $a, b \in L(X)$, then $a * b \in L(X)$ and $x * y \in V(a * b)$ for all $x \in V(a)$ and all $y \in V(b)$.*
(iii) *If $a, b \in L(X)$, then $a * x = a * b$ for all $x \in V(b)$.*
(iv) *$L(X)$ is a subalgebra of X.*
(v) *$a \in L(X)$ if and only if $x * (x * a) = a$ for all $x \in X$.*

For any non-empty subset A of X, we denote

$$L(A) = \{0 * (0 * x) : x \in A\}.$$

PROPOSITION 2 ([4]). *Let I be an ideal of a BCI-algebra X. Then*

(i) *$L(I) = I \cap L(X)$.*
(ii) *$L(I)$ is an ideal of $L(X)$.*
(iii) *I is closed if and only if $L(I)$ is closed in $L(X)$.*

PROPOSITION 3 ([4]). *Let X be a BCI-algebra and A a non-empty subset of $L(X)$. Denote $I = \bigcup_{a \in A} V(a)$. Then I is an ideal (resp. a subalgebra) of X if and only if A is an ideal (resp. a subalgebra) of $L(X)$.*

Let I be an ideal of a BCI-algebra X. By $CL(I)$ we denote the set $\bigcup_{a \in L(I)} V(a)$.

From Propositions 2 and 3 we have the following corollary.

COROLLARY 4. *Let I be an ideal of a BCI-algebra X. Then*

(i) *$CL(I)$ is an ideal of X.*

(ii) *$CL(I)$ is closed if and only if I is closed.*

A BCK-algebra X is said to be *positive implicative* [3] if $(x*z)*(y*z) = (x*y)*z$ for all $x, y, z \in X$. A BCI-algebra X is said to be *weakly positive implicative* [1] if $((x*z)*z)*(y*z) = (x*y)*z$ for all $x, y, z \in X$.

We propose a characterization of a weakly positive implicative BCI-algebra.

THEOREM 5. *Let X be a BCI-algebra. Then the following are equivalent:*

(i) *X is weakly positive implicative.*

(ii) $((x*y)*y)*(0*y) = x*y$.

(iii) $x*(x*y) = ((x*(x*y))*(x*y))*(0*(x*y))$.

(iv) $((x*z)*z)*(y*z) = 0$ *implies* $(x*y)*z = 0$.

(v) $((x*y)*y)*(0*y) = 0$ *implies* $x*y = 0$.

PROOF. (i)$\Leftrightarrow$(ii): See [1, Theorem 3].

(ii)$\Rightarrow$(iii): In (ii), if we substitute $x*y$ for y, then we have (iii).

(iii)$\Rightarrow$(ii): In (iii), substituting $x*y$ for y; then

$$x*(x*(x*y)) = ((x*(x*(x*y)))*(x*(x*y)))*(0*(x*(x*y))).$$

It follows from (3), (4), (5) and Proposition 1(v) that

$$\begin{aligned} x*y &= ((x*y)*(x*(x*y)))*((0*x)*((0*x)*(0*y))) \\ &= ((x*(x*(x*y)))*y)*(0*y) \\ &= ((x*y)*y)*(0*y). \end{aligned}$$

Hence (ii) holds.

(i) $\Rightarrow$(iv): It is an immediate consequence of weakly positive implicativity.

(iv) $\Rightarrow$(v): It is trivial.

(v) $\Rightarrow$(ii): Putting $u = ((x * y) * y) * (0 * y)$; then $u * (x * y) = 0$ and $(((x * u) * y) * y) * (0 * y) = (((x * y) * y) * (0 * y)) * u = 0$. It follows from (v) that $(x * y) * u = (x * u) * y = 0$. Hence $x * y = u$, and so $((x * y) * y) * (0 * y) = x * y$. Therefore (ii) holds. □

DEFINITION 1. A non-empty subset I of a BCI-algebra X is called a *weakly positive implicative ideal* of X if

(i) $0 \in I$,

(ii) $(((x * y) * y) * (0 * y)) * z \in I$ and $z \in I$ imply $x * y \in I$.

THEOREM 6. *Any weakly positive implicative ideal is an ideal.*

PROOF. Let I be a weakly positive implicative ideal of a BCI-algebra X and let $x, y \in X$ be such that $x * y \in I$ and $y \in I$. Then by using (2), we have $(((x * 0) * 0) * (0 * 0)) * y = x * y \in I$ and $y \in I$. It follows from Definition 1(ii) that $x = x * 0 \in I$. Therefore I is an ideal of X. □

The converse of Theorem 6 is not true, as seen in the next example.

EXAMPLE 1. Let $X = \{0, 1, 2, 3, 4, 5\}$. The binary operation $*$ is defined by the following table:

*	0	1	2	3	4	5
0	0	0	0	0	0	5
1	1	0	1	0	0	5
2	2	2	0	0	0	5
3	3	3	3	0	0	5
4	4	4	4	3	0	5
5	5	5	5	5	5	0

By routine calculations we know that X is a BCI-algebra, and $I = \{0, 1\}$ is an ideal of X but not weakly positive implicative, because

$$(((4 * 3) * 3) * (0 * 3)) * 1 = (3 * 3) * 1 = 0 \in I \quad \text{and}$$
$$1 \in I \quad \text{but} \quad 4 * 3 = 3 \notin I.$$

THEOREM 7. *Let I be an ideal of a BCI-algebra X. Then the following conditions are equivalent:*

(i) *I is weakly positive implicative.*

(ii) *$((x*y)*y)*(0*y) \in I$ implies $x*y \in I$.*

(iii) *$((x*z)*z)*(y*z) \in I$ implies $(x*y)*z \in I$.*

PROOF. (i)⇒(ii): Suppose I is a weakly positive implicative ideal of X and let $x, y \in X$. If $((x*y)*y)*(0*y) \in I$ then $(((x*y)*y)*(0*y))*0 \in I$ and $0 \in I$. It follows that $x*y \in I$.

(ii)⇒(iii): Suppose $((x*z)*z)*(y*z) \in I$ for $x, y, z \in X$. Putting $u = x*y$, then we have

$$\begin{aligned}((u*z)*z)*(0*z) &= (((x*y)*z)*z)*(0*z)\\ &= (((x*z)*z)*y)*((y*z)*y)\\ &\leq ((x*z)*z)*(y*z),\end{aligned}$$

and hence $((u*z)*z)*(0*z) \in I$. From (ii) it follows that $(x*y)*z = u*z \in I$.

(iii)⇒(i): Obviously $0 \in I$. Suppose $(((x*y)*y)*(0*y))*z \in I$ and $z \in I$ for $x, y, z \in X$. Then $((x*y)*y)*(0*y) \in I$ as I is an ideal of X. By (iii), we have $x*y = (x*0)*y \in I$. Therefore I is a weakly positive implicative idal of X. □

THEOREM 8. *Let I be a closed ideal of a BCI-algebra X. Then I is a weakly positive implicative ideal of $CL(I)$ if and only if for all $a \in CL(I)$, the set $A_a := \{x \in X : x*a \in I\}$ is an ideal of X.*

PROOF. Suppose I is a weakly positive implicative ideal of $CL(I)$ and let $a \in CL(I)$. Then clearly $0*a \in I$ or $0 \in A_a$. Let $x, y \in X$ be such that $x*y \in A_a$ and $y \in A_a$. Then $(x*y)*a, y*a \in I \subseteq CL(I)$, and hence $x, y \in CL(I)$ because $CL(I)$ is an ideal of X. Noticing from (3) and (6) that $((x*a)*a)*(y*a) \leq (x*y)*a$ and using (8); then $((x*a)*a)*(y*a) \in I$ and hence $(x*a)*a \in I$. Thus $((x*a)*a)*(0*a) \in I$ as I is a closed ideal and so a subalgebra. It follows from Theorem 7(ii) that $x*a \in I$ or $x \in A_a$. Therefore A_a is an ideal of X.

Conversely suppose that A_a is an ideal of X for all $a \in CL(I)$. If $((x*y)*y)*(0*y) \in I$ for $x, y \in CL(I)$, then $(x*y)*(0*y) \in A_y$.

By Proposition 2 we have $(0 * y) * y \in L(CL(I)) = L(I) \subseteq I$, and hence $y, 0 * y \in A_y$. It follows that $x \in A_y$ or $x * y \in I$. Therefore I is a weakly positive implicative ideal of $CL(I)$. □

COROLLARY 9. *Let I be a closed ideal of a BCI-algebra X and $L(X)$ a subset of I. Then I is a weakly positive implicative ideal of X if and only if for all $a \in X$, A_a is an ideal of X.*

COROLLARY 10. *Let I be a closed ideal of a BCI-algebra X. If I is a weakly positive implicative ideal of X, then for all $a \in CL(I)$, A_a is an ideal of X.*

THEOREM 11. *Let A and B be ideals of a BCI-algebra X such that $A \subseteq B$. If A is weakly positive implicative, then so is B.*

PROOF. Assume that A is a weakly positive implicative ideal of X. Let $x, y \in X$ be such that $((x * y) * y) * (0 * y) \in B$. Putting $u = ((x * y) * y) * (0 * y)$ and using (3); then we have $(((x * u) * y) * y) * (0 * y) = 0 \in A$. It follows from (3) and Theorem 7(ii) that $(x * y) * u = (x * u) * y \in A \subseteq B$, and hence $x * y \in B$. This proves that B is a weakly positive implicative ideal of X. □

Finally we give a characterization of weakly positive implicative BCI-algebras in terms of weakly positive implicative ideals.

THEOREM 12. *Let X be a BCI-algebra. Then the following are equivalent:*

(i) *X is weakly positive implicative.*
(ii) *$\{0\}$ is a weakly positive implicative ideal of X.*
(iii) *Every ideal of X is weakly positive implicative.*

PROOF. (i)⇒(ii): Suppose that X is weakly positive implicative. If $((x * y) * y) * (0 * y) \in \{0\}$ for $x, y \in X$, then $((x * y) * y) * (0 * y) = 0$. It follows from Theorem 5(v) that $x * y = 0 \in \{0\}$. Hence $\{0\}$ is a weakly positive implicative ideal of X.

(ii)⇒(iii): It is an immediate consequence of Theorem 11.

(iii)⇒(i): Assume that every ideal of X is weakly positive implicative. Setting $u = ((x * y) * y) * (0 * y)$ for all $x, y \in X$ and using (3); then we obtain $u * (x * y) = 0$ and $(((x * u) * y) * y) * (0 * y) = 0$. Since $\{0\}$ is a weakly positive implicative ideal of X, it follows from Theorem 7(ii) that $(x*y)*u = (x*u)*y = 0$, and hence $x*y = u = ((x*y)*y)*(0*y)$. Therefore X is weakly positive implicative. □

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