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ON THE SINGULAR VALUES OF THE QUOTIENT OF DRINFELD DISCRIMINANT FUCTIONS

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ABSTRACT. This paper is concerned with the prime factorization of the quotient of Drinfeld discriminant functions in an analytic way.

0. Introduction

The study of Drinfeld modules of rank 2 as an analogy with elliptic curves is an interesting subject. The prime factorization of the values $\Delta(\alpha z)/\Delta(z)$ for imaginary quadratic z, was done completely by Hasse in the number field case. In [3], the general tables of its prime factorization is given. The analogy in the function field case is also interesting. When $\mathfrak{o}$ is the full ring of integers of a global fuction field k, the results are given by Hayes using sgn-normalized Drinfeld module which has no counterpart in the classical case.(cf [6] 5.6 and [7] 4.18) In this article we get the same results for orders of imaginary quadratic function fields in an analytic way using Drinfeld modular functions.

1. Preliminaries

Let K be the rational function field $\mathbb{F}_q(T)$ over the finite field $\mathbb{F}_q$. Let K_∞ be the completion of K at $\infty = (1/T)$ and C the completion of the algebraic closure of K_∞. Put $A = \mathbb{F}_q[T]$.

An element

$$\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

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in $M_2(A)$, the set of 2×2 matrices with entries in A, is called *primitive* if g.c.d.$(a, b, c, d) = 1$. Let n be a monic polynomial in A. Define

$$\Delta_n^* = \{\alpha \in M_2(A) | \det \alpha = \mu n \text{ for some } \mu \in \mathbb{F}_q^*, \alpha \text{ is primitive}\}.$$

Then

$$\Gamma = GL_2(A) = \{\gamma \in M_2(A) | \det \gamma \in \mathbb{F}_q^*\}$$

acts on Δ_n^* by left or right multiplication. Following the classical method in [8], it is easy to see that Γ operates left transitively on the right Γ-cosets and also right transitively on the left Γ-cosets of Δ_n^*. In [2], the elements

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

in $M_2(A)$ with a and d monic, $ad = n$ and $\deg b < \deg d$ form distinct left Γ-coset representatives of Δ_n^*.

For any rank 2 A-lattice Λ, let ϕ^Λ be the Drinfeld module of rank 2 corresponding to Λ. For $z \in \Omega = C - K_\infty$, we also denote the Drinfeld module of rank 2 corresponding to the lattice $[1, z]$ by ϕ^z. Write

$$\phi_T^\Lambda = TX + g(\Lambda)X^q + \Delta(\Lambda)X^{q^2},$$

with $g(\Lambda), \Delta(\Lambda) \in C$. The j-invariant $j(\Lambda)$ is defined to be $g^{q+1}(\Lambda)/\Delta(\Lambda)$. In case $\Lambda = [z, 1]$, $g(\Lambda), \Delta(\Lambda)$ and $j(\Lambda)$ are denoted simply by $g(z), \Delta(z)$ and $j(z)$.

Let $L = \bar{\pi}A$ be the rank 1 A-lattice in C associated to the Carlitz module

$$\rho_T(X) = TX + X^q.$$

Put $t = t(z) = e_L^{-1}(\bar{\pi}z)$, where e_L is defined by

$$e_L(z) = z \prod_{\lambda \in L - \{0\}} (1 - z/\lambda).$$

By a *modular function* we mean a meromorphic function on Ω invariant under Γ and having t-expansions at infinity. Then j is a modular function and holomorphic on Ω. It can be shown that j is of the form

$$\frac{1}{s} + h(s),$$

where $s = t^{q-1}$ and h is a power series with coefficients in A, using the result in [4],(6.6),(6.7). Because the only modular functions holomorphic on both Ω and infinity are constants, we get

THEOREM 1.1. *Let f be a modular function which is holomorphic on Ω with a meromorphic s-expansion*

$$f = \sum c_i s^i.$$

Then f is a polynomial in j with coefficients in the module generated by c_i over A.

THEOREM 1.2 ([2], (2.5)). *If z is imaginary quadratic, that is $K(z)$ is a quadratic extension of K where ∞ does not split complitely, then $j(z)$ is integral over A.*

Let k be an imaginary quadratic extension field of K, and $\mathfrak{o}_k = [z, 1]$ be the ring of integers in k. By an *order* $\mathfrak{o}$ in k, we mean a subring of $\mathfrak{o}_k$ whose dimension over A is two. Let c be the unique monic polynomial in A such that $\mathfrak{o} \cap Az = Acz$. Then it can be shown that $\mathfrak{o} = [cz, 1]$. The number c is called the *conductor* of $\mathfrak{o}$. Let $\mathfrak{a}$ be an $\mathfrak{o}$-ideal. We shall say that $\mathfrak{a}$ is *prime* to c if either $\mathfrak{a} + c\mathfrak{o} = \mathfrak{o}$ or $\mathfrak{a} + c\mathfrak{o}_k = \mathfrak{o}$. The two conditions are actually equivalent. Following the classical method([8],p92), we have

THEOREM 1.3. *There is a multiplicative bijection between the monoid of $\mathfrak{o}_k$–ideals prime to c and the monoid of $\mathfrak{o}$-ideals prime to c. If we let $I_k(c)$ be the set of $\mathfrak{o}_k$-ideals prime to c, and $I_{\mathfrak{o}}(c)$ be the set of $\mathfrak{o}$-ideals prime to c, such inverse two mappings are given by*

$$\mathfrak{a} \to \mathfrak{a} \cap \mathfrak{o} \text{ for } \mathfrak{a} \in I_k(c) \text{ and}$$
$$\mathfrak{a} \to \mathfrak{a}\mathfrak{o}_k \text{ for } \mathfrak{a} \in I_{\mathfrak{o}}(c).$$

2. Integrality of $\Delta(\alpha z)/\Delta(z)$

In this section, we want to describe the values $\dfrac{\Delta(\alpha z)}{\Delta(z)}$ for imaginary quadratic z and $\alpha \in M_2(A)$. Define

$$\varphi_\alpha(\Lambda) = (\det \alpha)^{q^2-1} \frac{\Delta\left(\alpha \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right)}{\Delta\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right)},$$

for a given rank 2 A-lattice $\Lambda = [x_1, x_2]$. In case $\Lambda = [1, z]$, it will be also denoted by $\varphi_\alpha(z)$. For any constant $c \in A$, we have $\varphi_{c\alpha}(\Lambda) = c^{q^2-1}\varphi_\alpha(\Lambda)$, so it suffices to calculate its value for any primitive matrix α. We may assume that α is primitive. The multiplication by an element in Γ does not change the lattice Λ, so we may assume that α is a left Γ-coset representative of Δ_n^*. In the following, α is in triangular form

$$\alpha = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

with a and d monic, $ad = n$ and deg $b <$deg d, unless otherwise specified. Then

$$\varphi_\alpha(z) = (\det\ \alpha)^{q^2-1} d^{-(q^2-1)} \frac{\Delta(\alpha z)}{\Delta(z)} = a^{q^2-1}\frac{\Delta(\alpha z)}{\Delta(z)}.$$

Let t, $s = t^{q-1}$ and $\bar{\pi}$ be as in §1. Then the s-expansion for Δ is of the form

$$\Delta = \bar{\pi}^{q^2-1}\left(\sum_{i=1}^{\infty} a_i s^i\right)$$

with $a_1 = -1$ and $a_i \in A$.(e.g.[4])

Let B be a ring and $B((t_1, t_2, \ldots, t_M))$ be the ring of meromorphic power series with n-variables.

LEMMA 2.1. *Let*

$$\sum_{i=-N}^{\infty} a_i t_j^i \in B((t_j)), j = 1, \ldots, M$$

be given. Put

$$f(t_1, \ldots, t_M) = \prod_{j=1}^{M}\left(\sum_{i=-N}^{\infty} a_i t_j^i\right).$$

Write

$$f(t_1, \ldots, t_M) = \prod_{j=1}^{M} t_j^{-N} \sum_{i=0}^{\infty} f_i,$$

where f_i is the sum of monomials of $t_1, \ldots, t_M$ in f, with homogeneous degree i. Then f_i is a polynomial of the elementary symmetric polynomials of $t_1, \ldots, t_M$ with coefficients in B.

PROOF. We may assume that each $\sum_{i=-N}^{\infty} a_i t_j^i, j = 1, \ldots, M$ is a finite sum because each f_k is determined by $\prod_{j=1}^{M} \sum_{i=-N}^{k+MN} a_i t_j^i$. For any permutation σ of the integers $1, \ldots, M$, it is easy to see that

$$f_i(t_{\sigma(1)}, \ldots, t_{\sigma(M)}) = f_i(t_1, \ldots, t_M).$$

This proves our lemma.

Define the *a-th inverse cyclotomic polynomial* $f_a(X) \in A[X]$ for $a \in A$, not necessarily monic, by

$$f_a(X) = \rho_a(X^{-1})X^{|a|}$$

where $|a| = q^{\deg a}$. Then it is easy to see that the leading coefficient of a is the constant term of $f_a(X)$ and a is the leading coefficient of $f_a(X)$. In particular, if a is monic, the constant term of $f_a(X)$ is 1. For a monic element $n \in A$, let

$$S = \{\gamma | \gamma = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, a \in A, \deg a < \deg n\}.$$

Then the number of elements in S is $|n|$.

LEMMA 2.2. $\prod_{\gamma \in S} t(\frac{\gamma z}{n}) = t(z)$.

PROOF. Comparing the roots of polynomials, it is easy to see that

$$f_n(X) = n \prod_{\deg a < \deg n, a \neq 0} \left(X + \left(e_{\bar{\pi} A} \left(\frac{\bar{\pi} a}{n} \right) \right)^{-1} \right)$$

Since n is monic, the constant term of $f_n(X)$ is 1. Thus we have

$$n \left(\prod_{\deg a < \deg n, a \neq 0} \left(e_{\bar{\pi} A} \left(\frac{\bar{\pi} a}{n} \right) \right)^{-1} \right) = 1.$$

It follows that

$$f_n\left(t\left(\frac{z}{n}\right)\right) = n \prod_{\deg a<\deg n, a\neq 0} \left(t\left(\frac{z}{n}\right) + \left(e_{\bar{\pi}A}\left(\frac{\bar{\pi}a}{n}\right)\right)^{-1}\right)$$
$$= n \prod_{\deg a<\deg n, a\neq 0} \frac{e_{\bar{\pi}A}(\frac{\bar{\pi}a}{n})t(\frac{z}{n}) + 1}{e_{\bar{\pi}A}(\frac{\bar{\pi}a}{n})}$$
$$= \prod_{\deg a<\deg n, a\neq 0} \left(e_{\bar{\pi}A}\left(\frac{\bar{\pi}a}{n}\right) t\left(\frac{z}{n}\right) + 1\right).$$

It is easy to see that

$$t\left(\frac{z+a}{n}\right) = \frac{t(\frac{z}{n})}{1 + t(\frac{z}{n})e_{\bar{\pi}A}(\frac{\bar{\pi}a}{n})}$$

for any $a \in A$. Then

$$\prod_{\gamma\in S} t\left(\frac{\gamma z}{n}\right) = \prod_{\deg a<\deg n} t\left(\frac{z+a}{n}\right)$$
$$= t\left(\frac{z}{n}\right) \prod_{\deg a<\deg n, i\neq 0} \frac{t(\frac{z}{n})}{1 + t(\frac{z}{n})e_{\bar{\pi}A}(\frac{\bar{\pi}a}{n})}$$
$$= \frac{t(\frac{z}{n})^{|n|}}{f_n(t(\frac{z}{n}))}$$
$$= t(z).$$

LEMMA 2.3. *The elementary symmetric polynomials of* $t(\frac{\gamma z}{n}), \gamma \in S$, *lie in* $A[t(z)]$.

PROOF. Write

$$\rho_n(X) = nX + \sum_{i=1}^{\deg n} a_i X^{q^i}$$
$$= \prod_{\deg a<\deg n} \left(X + e_{\bar{\pi}A}\left(\frac{\bar{\pi}a}{n}\right)\right)$$

Then

$$\prod_{\gamma\in S}\left(X+e_{\bar\pi A}\left(\frac{\bar\pi\gamma z}{n}\right)\right)=\prod_{\deg a<\deg n}\left(X+e_{\bar\pi A}\left(\frac{\bar\pi z}{n}\right)+e_{\bar\pi A}\left(\frac{\bar\pi a}{n}\right)\right)$$
$$=\rho_n\left(X+e_{\bar\pi A}\left(\frac{\bar\pi z}{n}\right)\right)$$
$$=n\left(X+e_{\bar\pi A}\left(\frac{\bar\pi z}{n}\right)\right)$$
$$+\sum_{i=1}^{\deg n}a_i\left(X^{q^i}+\left(e_{\bar\pi A}\left(\frac{\bar\pi z}{n}\right)\right)^{q^i}\right).$$

Comparing the coefficients of X^k, we have that the symmetric polynomials of $e_{\bar\pi A}(\frac{\bar\pi\gamma z}{n}),\gamma\in S$, lie in A, except $\prod_{\gamma\in S}e_{\bar\pi A}(\frac{\bar\pi\gamma z}{n})=1/t(z)$(e.g. Lemma1.1). Since $t(\frac{\gamma z}{n})=e_{\bar\pi A}(\frac{\bar\pi\gamma z}{n})^{-1}$, it is easy to see that our lemma holds.

THEOREM 2.4. *The function φ_α is integral over $A[\Lambda_n][j]$, where Λ_n is the set of all n-torsion points of the Calitz module ρ.*

PROOF. Let $\psi(n)$ be the number of representatives of the left cosets of primitive matrices in $M_2(A)$ and let $\alpha_1,\ldots,\alpha_{\psi(n)}$ be representatives. Since $\varphi_\alpha(\gamma z)=\varphi_{\alpha\gamma}(z)$ for $\gamma\in GL_2(A)$ and γ permutes the coset representatives α_i, any symmetric function of φ_{α_i} is fixed by the elements in $GL_2(A)$. Since Δ does not vanish on Ω, $\varphi_{\alpha_i}(z)$ is holomorphic on Ω. The $t(\frac{z}{n})$-expansion of $t(\frac{az+b}{d})$ is given by

$$t\left(\frac{az+b}{d}\right)=\frac{t(\frac{a^2z}{n})}{1+e_{\bar\pi A}(\frac{\bar\pi ab}{n})t(\frac{a^2z}{n})}$$
$$=\frac{t(\frac{z}{n})^{|a^2|}}{f_{a^2}(t(\frac{z}{n}))+e_{\bar\pi A}(\frac{\bar\pi ab}{n})t(\frac{z}{n})^{|a^2|}}\in A[\Lambda_n][[t(\frac{z}{n})]].$$

Note that the constant term in the denominator of above equation lies in $\mathbb{F}_q^*$, so it has the inverse in $A[\Lambda_n][[t(\frac{z}{n})]]$. We also have the following $t(\frac{z}{n})$-expansion of

$$t(z)=\frac{t(\frac{z}{n})^{|n|}}{f_n(t(\frac{z}{n}))}\in A[[t(\frac{z}{n})]].$$

Since n is monic, the constant term of $t(z)$ in the $t(\frac{z}{n})$-expansion is 1. Put

$$\Delta^*(z) =: \bar{\pi}^{1-q^2}\Delta(z) = \sum_{i=q-1}^{\infty} a_i t^i,$$

with $a_i \in A, a_{q-1} = -1$. Then

$$\begin{aligned}\varphi_\alpha(z) &= a^{q^2-1}\frac{\Delta^*(\frac{az+b}{d})}{\Delta^*(z)} \\ &= a^{q^2-1}\frac{\sum_{i=q-1}^{\infty} a_i t(\frac{az+b}{d})^i}{\sum_{i=q-1}^{\infty} a_i t(z)^i}.\end{aligned}$$

Thus $\varphi_\alpha(z)$ has a $t(\frac{z}{n})$-expansion with coefficients in $A[\Lambda_n]$. Let f be any symmetric polynomial of $\varphi_{\alpha_i}(z), i = 1, \ldots, \psi(n)$. Then f is fixed by $GL_2(A)$, f is holomorphic on Ω and f has a meromorphic $t(\frac{z}{n})$-expansion with coefficients in $A[\Lambda_n]$. Write

$$f = \sum c_i t\left(\frac{z}{n}\right)^i,$$

with $c_i \in A[\Lambda_n]$. Then

$$f^{|n|} = \prod_{\gamma \in S} f(\gamma z) = \prod_{\gamma \in S}\left(\sum c_i t\left(\frac{\gamma z}{n}\right)^i\right).$$

By Lemma 2.1 and Lemma 2.3, $f^{|n|}$ can be written as

$$\sum d_i t(z)^i,$$

with $d_i \in A[\Lambda_n]$. Since $f^{|n|}$ is fixed by

$$\begin{pmatrix} 1 & 0 \\ 0 & \gamma \end{pmatrix}, \gamma \in \mathbb{F}_q^*,$$

$d_i = 0$ if i is not divisible by $q-1$. Thus $f^{|n|}$ has a meromorphic s-expansion. Then $f^{|n|}$ lies in $A[\lambda_n][j]$ by Theorem 1.1, so we have

$$\left(\prod_{i=1}^{\psi(n)}(X - \varphi_{\alpha_i}(z))\right)^{|n|} \in A[\Lambda_n][j][X].$$

This completes the proof.

Applying Theorem 1.2 and Therem 2.4, we have

THEOREM 2.5. *For imaginary quadratic z, the values $\varphi_\alpha(z)$ are integral over A.*

Note that Theorem 2.4 and Theorem 2.5 hold even if α is not primitive.

3. Prime factorization of $\varphi_\alpha(z)$

In this section, we want to describe the prime factorization of the values $\varphi_\alpha(z)$ for imaginary quadratic z.

THEOREM 3.1. *For imaginary quadratic z, the value $\varphi_\alpha(z)$ is integral and divides $(\det\alpha)^{q^2-1}$.*

PROOF. It suffices to check the divisibility. Let $\alpha' = n\alpha^{-1}$ with $n = \det\alpha$. Then $\det\alpha' = n$. Since $\alpha\alpha' = nI$ and Δ is a modular form of weight q^2-1, we have

$$\Delta\left(\alpha\alpha'\begin{pmatrix} z \\ 1\end{pmatrix}\right) = n^{-(q^2-1)}\Delta\left(\begin{pmatrix} z \\ 1\end{pmatrix}\right).$$

Then

$$\begin{aligned}
&\varphi_{\alpha'}\left(\alpha\begin{pmatrix} z \\ 1\end{pmatrix}\right)\varphi_\alpha\left(\begin{pmatrix} z \\ 1\end{pmatrix}\right) \\
&= (\det\alpha')^{q^2-1}\frac{\Delta\left(\alpha'\alpha\begin{pmatrix} z \\ 1\end{pmatrix}\right)}{\Delta\left(\alpha\begin{pmatrix} z \\ 1\end{pmatrix}\right)}(\det\alpha)^{q^2-1}\frac{\Delta\left(\alpha\begin{pmatrix} z \\ 1\end{pmatrix}\right)}{\Delta\left(\begin{pmatrix} z \\ 1\end{pmatrix}\right)} \\
&= n^{q^2-1} \\
&= (\det\alpha)^{q^2-1}.
\end{aligned}$$

Since both $\varphi_{\alpha'}\left(\alpha\begin{pmatrix} z \\ 1\end{pmatrix}\right)$ and $\varphi_\alpha\left(\begin{pmatrix} z \\ 1\end{pmatrix}\right)$ are integral over A, we get the result.

By an *algebraic integer*, we mean an element in some algebraic extension field over K which is integral over A. We use the following notations.

If ξ is an algebraic integer and $\mathfrak{a}$ is an ideal in the ring of the integers of an algebraic extension field M of K, we write $\xi \approx \mathfrak{a}$ to mean that $\xi\mathfrak{o}_L = \mathfrak{a}\mathfrak{o}_L$ in some algebraic extension field L of M. We then say that ξ and $\mathfrak{a}$ are *associated*. Similarly if ξ_1 and ξ_2 are algebraic integers, we write $\xi_1 \approx \xi_2$ to mean that ξ_1/ξ_2 is a unit. In this case, we say that ξ_1 and ξ_2 are *associated*.

Let $\mathfrak{a}$ be a proper $\mathfrak{o}$-ideal. Define $\mathbb{N}\mathfrak{a}$ to be the unique monic generator of the ideal generated by $\{Norm_{k/K}a | a \in \mathfrak{a}\}$. Then we have

$$q^{\deg(\mathbb{N}\mathfrak{a})} = (\mathfrak{o} : \mathfrak{a}).$$

If $\mathfrak{b} = [z_1, z_2]$ is another proper $\mathfrak{o}$-ideal, then we can find $\alpha \in M_2(A)$ such that $\alpha \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ is a basis of $\mathfrak{ab}$. Since α is unique up to multiplication by an element in $GL_2(A)$, the following definition is well-defined.

$$\varphi_{\mathfrak{a}}(\mathfrak{b}) =: \varphi_\alpha\left(\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right) = (\mathbb{N}\mathfrak{a})^{q^2-1}\frac{\Delta(\mathfrak{ab})}{\Delta(\mathfrak{b})}$$

THEOREM 3.2. *Let $p(T)$ be a monic prime element in A which splits completely in k and dose not divide the conductor c of $\mathfrak{o}$. Let $p(T)\mathfrak{o} = \mathfrak{p}\mathfrak{p}'$ be the prime factorization in $\mathfrak{o}$. Then*

$$\varphi_{\mathfrak{p}}(\mathfrak{a}) = p(T)^{q^2-1}\frac{\Delta(\mathfrak{pa})}{\Delta(\mathfrak{a})} \approx \mathfrak{p}'^{q^2-1}.$$

Note that by Theorem 1.3, both $\mathfrak{p}$ and $\mathfrak{p}'$ can be viewed as ideals in the ring of integers $\mathfrak{o}_k$.

PROOF. Choose $\lambda \in \mathfrak{p} - \mathfrak{p}^2$ prime to c such that λ is not contained in $\mathfrak{p}'$. Put $\mathfrak{b} = \lambda\mathfrak{p}^{-1}$. Then $\mathfrak{b}$ is a proper $\mathfrak{o}$-ideal prime to $p(T)$ such that $\mathfrak{bp} = \lambda\mathfrak{o}$ is principal. Then

$$\begin{aligned}
\varphi_{\mathfrak{b}}(\mathfrak{pa})\varphi_{\mathfrak{p}}(\mathfrak{a}) &= \mathbb{N}\mathfrak{b}^{q^2-1}\frac{\Delta(\mathfrak{bap})}{\Delta(\mathfrak{pa})}p(T)^{q^2-1}\frac{\Delta(\mathfrak{pa})}{\Delta(\mathfrak{a})} \\
&= (\mathbb{N}\mathfrak{b}p(T))^{q^2-1}\frac{\Delta(\lambda\mathfrak{a})}{\Delta(\mathfrak{a})} \\
&= \left(\frac{\mathbb{N}\mathfrak{b}p(T)}{\lambda}\right)^{q^2-1}.
\end{aligned}$$

By Theorem 3.1, $\varphi_{\mathfrak{b}}(\mathfrak{p}\mathfrak{a})$ is an algebraic integer dividing $\mathbb{N}\mathfrak{b}^{q^2-1}$ which is prime to $p(T)$, and $\varphi_{\mathfrak{p}}(\mathfrak{a})$ is an algebraic integer dividing $p(T)^{q^2-1}$. Therefore $\varphi_{\mathfrak{p}}(\mathfrak{a})$ is associated to the product of the prime factors of $\mathfrak{p}, \mathfrak{p}'$ which divide $(\frac{\mathbb{N}\mathfrak{b}p(T)}{\lambda})^{q^2-1}$. This proves our theorem.

COROLLARY 3.3. *Let $p(T)$ be a monic prime element in A such that its prime factorization in k is $\mathfrak{p}^2$. Suppose that $p(T)$ does not divide the conductor c of $\mathfrak{o}$. Then $\varphi_{\mathfrak{a}}(\mathfrak{p}) \approx \mathfrak{p}^{q^2-1}$ for any proper $\mathfrak{o}$-ideal $\mathfrak{a}$.*

PROOF. Choose $\lambda \in \mathfrak{p} - \mathfrak{p}^2$ prime to c. The rest are exactly the same as the proof of above theorem.

We know that representative matrices of $\Gamma\backslash\Delta^*_{p(T)}$ can be selected as

$$\alpha_{p(T)} = \begin{pmatrix} p(T) & 0 \\ 0 & 1 \end{pmatrix}$$

$$\alpha_i = \begin{pmatrix} 1 & a_i \\ 0 & p(T) \end{pmatrix},$$

with $\deg a_i < \deg p(T)$, $i = 2, 3, \ldots, \psi(P(T))$.

LEMMA 3.4. *Let $p(T)$ be a monic irreducible polynomial in A, and let z be imaginary quadratic. Then*

$$f(z) =: \varphi_{\alpha_{p(T)}}(z) \prod_i \varphi_{\alpha_i}(z) = p(T)^{q^2-1}.$$

PROOF. The s-expansion for φ_{α_i} is given by

$$\varphi_{\alpha_i} = \frac{\sum_{k=1}^{\infty} a_k s(\frac{z+a_i}{p(T)})^k}{\sum_{k=1}^{\infty} a_k s(z)^k}$$

with $a_k \in A$ and $a_1 = -1$. The s-expansion for $\varphi_{\alpha_{p(T)}}$ is given by

$$\varphi_{\alpha_{p(T)}} = p(T)^{q^2-1} \frac{\sum_{k=1}^{\infty} a_k s(p(T)z)^k}{\sum_{k=1}^{\infty} a_k s(z)^k}.$$

We have the following $t(\frac{z}{p(T)})$ expansions.

$$t\left(\frac{z+a_i}{p(T)}\right) = \sum_{k=0}^{\infty}(-1)^k\left(e_L\left(\frac{a_i\bar{\pi}}{p(T)}\right)^k t\left(\frac{z}{p(T)}\right)^{k+1}\right)$$

$$= t\left(\frac{z}{p(T)}\right) + \text{higher terms.}$$

$$t(p(T)z) = \frac{t(\frac{z}{p(T)})^{|p(T)^2|}}{f_{p(T)^2}(t(\frac{z}{p(T)}))} = t\left(\frac{z}{p(T)}\right)^{|p(T)^2|} + \text{higher terms.}$$

$$t(z) = \frac{t(\frac{z}{p(T)})^{|p(T)|}}{f_{p(T)}(t(\frac{z}{p(T)}))} = t\left(\frac{z}{p(T)}\right)^{|p(T)|} + \text{higher terms.}$$

Since $s = t^{q-1}$, the leading term of φ_{a_i} is given by

$$\frac{(-1)t\left(\frac{z}{p(T)}\right)^{q-1}}{(-1)t\left(\frac{z}{p(T)}\right)^{|p(T)|(q-1)}} = t\left(\frac{z}{p(T)}\right)^{(q-1)(1-|p(T)|)}$$

The leading term of $\varphi_{a_{p(T)}}$ is

$$p(T)^{q^2-1}\frac{(-1)t\left(\frac{z}{p(T)}\right)^{|p(T)^2|(q-1)}}{(-1)t\left(\frac{z}{p(T)}\right)^{|p(T)|(q-1)}} = p(T)^{q^2-1}t\left(\frac{z}{p(T)}\right)^{(q-1)(|p(T)^2|-|p(T)|)}$$

Hence the leading term of $\varphi_{a_{p(T)}}(z)\prod_i \varphi_{a_i}(z)$ is

$$\left(t\left(\frac{z}{p(T)}\right)^{(q-1)(1-|p(T)|)}\right)^{|p(T)|} p(T)^{q^2-1}t\left(\frac{z}{p(T)}\right)^{(q-1)(|p(T)^2|-|p(T)|)}$$
$$=p(T)^{q^2-1},$$

which is a constant. Following the proof of Theorem 2.4, $f(z)^{|p(T)|}$ lies in $A[\Lambda_{p(T)}][j]$ and it has a holomophic $t(z)$-expansion. Thus it shoud be constant and this completes the proof.

THEOREM 3.5. *Let $p(T), \mathfrak{o}, \mathfrak{a} = [z_1, z_2]$ and $p(T)\mathfrak{o} = \mathfrak{p}\mathfrak{p}'$ be as in Theorem 3.2. Let β_1 and β_2 be matrices of determinant $p(T)$ such that $\beta_1 \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ and $\beta_2 \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ are bases of $\mathfrak{p}\mathfrak{a}$ and $\mathfrak{p}'\mathfrak{a}$ respectively. If $\alpha \in M_2(A)$ has determinant $p(T)$ and α does not lie in the orbit of β_1 or β_2 under $\Gamma = GL_2(A)$, then $\varphi_\alpha \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ is a unit.*

PROOF. It is easy to see that if any integral matrix has determinant $p(T)$, then it is Γ-equivalent to a representive for left cosets of Γ in $\Delta^*_{p(T)}$. Since

$$\varphi_{\beta_1} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \varphi_{\beta_2} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \approx \mathfrak{p}'^{q^2-1} \mathfrak{p}^{q^2-1} = (p(T)^{q^2-1}),$$

all other terms in the product of lemma 3.4 cannot have prime factors. Thus $\varphi_\alpha \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ is a unit.

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