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SOME PROPERTIES OF VERMA MODULES OVER AFFINE LIE ALGEBRAS

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ABSTRACT. For nonintegrable weight $-\rho$, some weight multiplicities of the irreducible module $L(-\rho)$ over $A_{(1)}^{(1)}$ affine Lie algebras are expressed in terms of the colored partition functions. Also we find the multiplicity of $L(-\rho)$ in the Verma module $M(-\rho)$ for any affine Lie algebras.

1. Introduction

For integrable highest weight modules over symmetrizable Kac-Moody algebras, a character formula called the Weyl-Kac character formula, was obtained by Kac. The Weyl-Kac character formula applied to certain modules gives Lie-algebraic proofs of some classical identities like Jacobi's triple product identity.

In [5], a conjecture on the character formula of the nonintegrable highest weight module $L(-\rho)$ for affine Kac-Moody algebras i.e., the irreducible quotient of the Verma module $M(-\rho)$, was proved. And also some properties concerning embeddings of $M(-\rho)$ were given.

In this note we first consider an affine Kac-Moody algebra $G(A)$ whose corresponding Cartan matrix A is

$$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

We say that this affine Kac-Moody Lie algebra is of type $A_{(1)}^{(1)}$. And we present a concrete form of the character formula of the irreducible

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quotient $L(-\rho)$ over an affine Lie algebra of this type, where weight multiplicities are expressed in terms of the colored partition functions. We use the character formula in [5] and the explicit expressions of generalized partition functions proved by Kac and Peterson in [4].

Secondly we consider some Verma Modules over any affine Kac-Moody Lie algebra and we describe multiplicities of the irreducible quotients that occur in the Jordan-Hölder series of these Verma modules.

Terminologies and notations in this note are standard and they may be found in [2].

2. Preliminaries

Let $G(A)$ be a Kac-Moody Lie algebra associated to a symmetrizable generalized Cartan matrix A. Then $G(A)$ has a root space decomposition $G(A) = H \oplus_{\alpha\in\Delta} G_\alpha$, where $\Delta \in H^*$ is the root system which is equipped with a basis Π . Elements in Π are called simple roots. We denote the set of positive real(respectively, imaginary) roots by Δ^+_{re} (respectively, Δ^+_{im}). There exists an invariant bilinear form $(\cdot,\cdot)$ on $G(A)$ which is nondegenerate on H. And this induces a nondegenerate bilinear form $(\cdot,\cdot)$ on H^*.

For $\lambda \in H^*$, let $M(\lambda)$ be the Verma module of G with highest weight λ. The character of $M(\lambda)$ is defined to be the formal sum

$$ch\ M(\lambda) = \sum_{\mu\in H^*} \dim M(\lambda)_\mu e^\mu,$$

where $M(\lambda)_\mu = \{ v \mid h\cdot v = \mu(h)v, \text{ for all } h \in H \}$. When $M(\lambda)_\mu \neq \emptyset$, we call μ a weight of $M(\lambda)$ and $\dim M(\lambda)_\mu$ the weight multiplicity of the weight μ. The following is well known:

$$\begin{aligned} ch\ M(\lambda) &= e^\lambda \sum K(\lambda-\mu)e^{-\mu} \\ &= e^\lambda \prod_{\alpha\in\Delta_+} (1-e^{-\alpha})^{-\dim G_\alpha}, \end{aligned}$$

here K is the generalized partition function (see [2]).

The Verma module $M(\lambda)$ has a unique maximal submodule and the corresponding irreducible quotient is denoted as $L(\lambda)$. And the character of $L(\lambda)$ is defined in the same way as for Verma module $M(\lambda)$.

Let $\mu \in H^*$. A local composition series of Verma module $M(\lambda)$ at μ is a decreasing filtration $M(\lambda) = M_0 \supset M_1 \supset \cdots \supset M_t = 0$ of submodules of $M(\lambda)$ such that either $M_i/M_{i+1} \cong L(\beta)$ for some $\beta \geq \mu$ or $(M_i/M_{i+1})_\beta = 0$ for all $\beta \geq \mu$. The number of irreducible quotients isomorphic to $L(\mu)$ appearing in a local composition series of $M(\lambda)$ at μ is called the multiplicity of $L(\mu)$ in $M(\lambda)$. We denote this multiplicity by $[\, M(\lambda) : L(\mu)\,]$ (see [1]).

Given a generalized Cartan matrix A in case there is a nonzero column vector u of nonnegative integers such that $Au = 0$, the corresponding Lie algebra is called an affine Kac-Moody Lie Algebra. For an affine Kac-Moody Lie algebra every positive imaginary root is an integer multiple of the smallest positive imaginary root, which is denoted as δ.

Now, we fix a linear function $\rho \in H^*$ such that $2(\rho, \alpha) = (\alpha, \alpha)$ for all $\alpha \in \Pi$. The following results were proved by J. M. Ku (see [5]).

PROPOSITION 2.1. *Let $G(A)$ be an affine Kac-Moody Lie Algebra. Then the character formula of the irreducible $G(A)$ module $L(-\rho)$ over G is given by:*

$$ch\ L(-\rho) \;=\; e^{-\rho} \prod_{\alpha \in \Delta_+^{re}} (1 - e^{-\alpha})^{-\dim G_\alpha}.$$

PROPOSITION 2.2. *Let $G(A)$ be an affine Kac-Moody Lie algebra. Then*

$$[\, M(-\rho) \;:\; L(-\rho - n\delta)\,] \;=\; dim\ Hom_G\ (\, M(-\rho - n\delta)\ ,\ M(-\rho)\,).$$

3. Main Theorem

In this section, we first consider an affine Lie algebra $G(A)$ of type $A_{(1)}^{(1)}$. There are two simple roots α_1, α_2 and the set of roots are described:

$$\Delta_{re}^+ = \{ j\alpha_1 + (j+1)\alpha_2 \quad and \quad (j+1)\alpha_1 + j\alpha_2\ ,\ j = 0, 1, 2, \cdots \ \},$$

$$\Delta_{im}^- = \{ j\alpha_1 + j\alpha_2\ ,\ j = 0, 1, 2, \cdots \ \}.$$

All root spaces are one dimensional. The smallest positive imaginary root δ is $\alpha_1+\alpha_2$.

In the next theorem we show that the character of the irreducible module $L(-\rho)$ is linked to colored partition functions. First we recall the definition of the Euler function $\phi(q)$:

$$\phi(q) = \prod_{n\geq 1}(1-q^n) = \sum_{n\in Z}(-1)^n q^{\frac{1}{2}n(3n+1)}.$$

And the colored partition functions are functions defined to be functions $p^{(l)}(n)$ described in the following identity:

$$\sum_{n\geq 1} p^{(l)}(n)q^n = \prod_{n\geq 1}(1-q^n)^{-l}.$$

THEOREM 3.1. *Let $G(A)$ be an affine Lie algebra of type $A_{(1)}^{(1)}$ and let $L(-\rho)$ be the irreducible $G(A)$ module. Then*

$$dim\ L(-\rho)_{-\rho-l\delta} = \sum_{m+\frac{3n^2+n}{2}=l}\ \sum_{j\geq 0}(-1)^n(-1)^j p^{(3)}(m-\frac{1}{2}j(j+1)).$$

PROOF. Since $\dim G_\alpha = 1$ for any root $\alpha\in\Delta$, by Proposition 2.1 we have

$$ch\ M(-\rho) = e^{-\rho}\prod_{\alpha\in\Delta+}(1-e^{-\alpha})^{-\dim\ G_\alpha} = ch\ L(-\rho)\prod_{n\geq 1}(1-e^{-n\delta})^{-1}.$$

Here we denote $e^{-\delta}$ by q. Applying the Euler function and the generalized partition function K on $ch\ M(-\rho)$ we deduce the following:

$$\begin{aligned} ch\ L(-\rho) &= e^{-\rho}\sum_{n\in Z}(-1)^n q^{\frac{1}{2}n(3n+1)}\ ch\ M(-\rho) \\ &= e^{-\rho}\sum_{n\in Z}(-1)^n q^{\frac{1}{2}n(3n+1)}\sum_{a,b\geq 0} K(a\alpha_1+b\alpha_2)e^{-a\alpha_1}e^{-b\alpha_2}. \end{aligned}$$

Hence Proposition 5.9 in [4] implies

$$ch\ L(-\rho) = e^{-\rho}\sum_{n\in Z}(-1)^n q^{\frac{1}{2}n(3n+1)} \times$$
$$\sum_{j\geq 0}(-1)^j p^{(3)}((j+1)a - jb - \frac{1}{2}j(j+1))e^{-a\alpha_1}e^{-b\alpha_2}.$$

The dimension of the weight space $L(-\rho)_{-\rho-l\delta}$ is the coefficient of q^l in the right-hand side. Therefore, collecting up coefficients of q^m and q^n satisfying $m + \frac{3n^2+n}{2} = l$ we obtain the result.

COROLLARY 3.2. *Let* $L(-\rho)$ *be the irreducible* $G(A)$ *module and* $G(A)$ *be of type* $A^{(1)}_{(1)}$. *Then*

$$dim\ L(-\rho)_{-\rho-l\delta} = \sum_{n+\frac{k(k+1)}{2}=l} p^{(2)}(n)(-1)^k.$$

PROOF. Definition of the generalized partition function yields:

$$\phi(q)^3\sum_{m\geq 0} K(m\alpha_1 + m\alpha_2)q^m$$
$$= \sum_{k\geq 0}(-1)^k\phi(q)^3\sum_{m\geq 0} p^{(3)}(m - \frac{k(k+1)}{2})q^m$$
$$= \sum_{k\geq 0}(-1)^k q^{\frac{k(k+1)}{2}}.$$

Applying this to Theorem 3.1 gives the result.

REMARK. One can apply this process for affine Lie algebra of type $A^{(2)}_{(2)}$ to find a property similar to Corollary 3.2.

For the rest of this note we consider modules over any affine Kac-Moody Lie Algebra.

LEMMA 3.3. *For any affine Kac-Moody Lie algebra and positive integers* n, m *with* $n \geq m$, *we have*

$$[\ M(-\rho - m\delta)\ :\ L(-\rho - n\delta)\ ] \neq 0$$

and

$$[\ M(-\rho - m\delta)\ :\ L(-\rho - n\delta)\ ] = [\ M(-\rho)\ :\ L(-\rho - (n-m)\delta)\ ].$$

PROOF. The first assertion follows from Kac and Kazhdan's criterion in [4] since $(\delta, \delta) < o$. The second is obtained immediately from the following observation:

$$ch\ M(-\rho - m\delta) = e^{-m\delta} ch\ M(-\rho)$$
$$ch\ L(-\rho - n\delta) = e^{-n\delta} ch\ L(-\rho).$$

THEOREM 3.4. *For any affine Kac-Moody Lie algebra and nonnegative integers* n, m *with* $n \geq m$,

$$[\ M(-\rho - m\delta)\ :\ L(-\rho - n\delta)\ ] \geq \dim G_\delta$$

and the equality holds when $n = m + 1$.

PROOF. By Lemma 3.3 it is enough to prove for the case $m = 0$. Since δ is the smallest imaginary root, $r\delta$ can not be a positive root for $0 < r < 1$. Therefore, by proposition 4.1 in [3] we have

$$\dim Hom_G(\ M(-\rho - n\delta)\ ,\ M(-\rho)\ ) \geq \dim G_\delta.$$

This together with Corollary 5.6 in [5] completes the proof for the case $m = 0$.

Now let $n = 1$ and let β be a positive root. If $t\beta < \delta$ for some positive integer t then β cannot be an imaginary root, thus $t(\beta, \beta) > 0$. Again, by Proposition 4.1 in [3] together with Corollary 5.6 in [5] we obtain the equality.

REMARK. Proposition 2.1 is equivalent to the following:

$$\sum_{n \geq 0} [\ M(-\rho)\ :\ L(-\rho - n\delta)\ ] e^{-n\delta} = \prod_{n \geq 1} (1 - e^{-n\delta})^{-\dim G_{n\delta}}$$

Considering a nontwisted affine algebra $X_l^{(1)}$, by Theorem 3.4 we obtain

$$p^{(l)}(n) \geq l, \quad \text{for any integer} \quad n,$$
$$p^{(l)}(1) = l.$$

References

1. V. V. Deodhar, O. Gabber, and V. G. Kac, *Structure of some categories of representations of infinite dimensional Lie algebras*, Advances in Math. **45** (1982), 92-116.
2. V. Kac,, *Infinite dimensional Lie algebras*, Birkhäuser, Boston, Mass., 1983.
3. V. Kac and D. A. Kazhdan, *Structure of representations with highest weight of infinite dimensional Lie algebras*, Advances in Math. **34** (1979), 97-108.
4. V. Kac and D. H. Peterson, *Infinite-dimensional Lie algebras, theta functions and modula forms*, Advances in Math. **53** (1984), 125-264.
5. J. M. Ku, *Structure of Verma module* $M(-\rho)$ *over euclidean Lie algebras*, J. Algebra **124** (1989), 367-387.

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