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A SELECTION THEOREM AND ITS APPLICATION

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ABSTRACT. In this paper, we give equivalent forms of the selection theorem of Ding-Kim-Tan. As applications of the selection theorem of Ding-Kim-Tan, we obtain a fixed point theroem of Gale and Mas-Colell type and establish an equilibrium existence theorem for a qualitative game under suitable assumptions in a locally convex Hausdorff topological vector space.

1. Introduction

Selection theorem was firstly proved by Michael [9]. This theorem plays very important roles in nonlinear analysis [1,7,8,9,10,11].

Yannelis-Prabhakar [11] proved another selection theorem and obtained a fixed point theorem on the paracompact setting. Using their fixed point theorem, they proved an equilibrium existence theorem for a compact abstract economy.

Recently, Tarafdar [10] proved some selection theorem and obtained a fixed point theorem on an H-space under the compact assumption. He considered the abstract economy in which the commodity space is an H-space and proved by means of his fixed point theorem the existence of equilibrium points of such abstract economies. The H-space is a topological space equipped with the family of its nonempty contractible subsets. The conception of the H-space was firstly considered by Horvath

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[6]. Ding-Kim-Tan [1] gave an improved version of the selection theorem of Yannelis-Prabhakar [11] and a fixed point theorem on the paracompact setting. As applications of their fixed point theorem, they obtained new equilibrium existence theorems.

In this note, motivated by recent results in [1,10], we give equivalent forms of the selection theorem of Ding-Kim-Tan [1]. As applications of this selection theorem, we obtain a fixed point theorem of Gale and Mas-Colell [3] type and establish an equilibrium existence theorem for a qualitative game under suitable assumptions in a locally convex Hausdorff topological vector space.

2. A selection theorem

First, we give the relationships among several kinds of correspondences.

PROPOSITION 2.1. *Let X, Y be topological spaces and $T : X \to 2^Y$ a correspondence.*

(a) *T has an open graph, i.e., graph $T := \{(x, y) : y \in T(x)\}$ is open in $X \times Y$.*
(b) *T has open lower sections, i.e., for each $y \in Y$, $T^{-1}(y) := \{x \in X : y \in T(x)\}$ is open in X.*
(c) *for each $y \in Y$ with $T^{-1}(y) \neq \emptyset$, there exists a nonempty open subset O_y of X such that $O_y \subset T^{-1}(y)$.*
(d) *T is lower semicontinuous, i.e., for any $x \in X$ and any open subset G of Y with $T(x) \cap G \neq \emptyset$, there exists an open neighborhood $N(x)$ of x in X such that for each $z \in N(x)$, $T(z) \cap G \neq \emptyset$.*

Then we have
(1) (a) *implies* (b), (b) *implies* (c) *and* (b) *implies* (d).
(2) (c) *does not imply* (b).
(3) (c) *does not imply* (d).
(4) (d) *does not imply* (c).

PROOF. (1) It is clear that (a) implies (b), and (b) implies (c). By Proposition 4.1 in [11], we can see that (b) implies (d).
(2) Define a correspondence $T : R \to 2^R$ by for any $x \in R$, $T(x) = [x, x + 1]$. Then (c) holds for T but (b) does not hold for T.

(3) Define a correspondence $T : R \to 2^R$ by

$$T(x) = \begin{cases} [0,1] & \text{if } |x| < 1 \\ \{1\} & \text{if } x = \pm 1 \\ \{0\} & \text{if } |x| > 1. \end{cases}$$

Then (c) holds for T but (d) does not hold for T.
(4) Define a correspondence $T : R \to 2^R$ by for any $x \in R$, $T(x) = \{x^3\}$. Then (d) holds for T but (c) does not hold for T.

Our results of this paper are mainly concerned with the condition (c).

The following is the selection theorem of Ding-Kim-Tan [1], which generalizes one of Yannelis-Prabhakar [11, Theorem 3.1].

THEOREM 2.1. *Let X be a nonempty paracompact subset of a Hausdorff topological space and Y a nonempty convex subset of a Hausdorff topological vecotr space. Suppose that $S, T : X \to 2^Y$ are correspondences such that*

(1) *for each $x \in X$, $coS(x) \subset T(x)$ and $S(x) \neq \emptyset$, where $coS(x)$ is the convex hull of the set $S(x)$,*
(2) *for each $y \in Y$, $S^{-1}(y)$ is open in X.*

Then T has a continuous selection, i.e., there exists a continuous function $f : X \to Y$ such that for each $x \in X$, $f(x) \in T(x)$.

Now we give an equivalent form of Theorem 2.1 as follows;

THEOREM 2.2. *Let X be a nonempty paracompact subset of a Hausdorff topological space and Y a nonempty convex subset of a Hausdorff topological vector space. Suppose that $S, T : X \to 2^Y$ are correspondences such that*

(1) *for each $x \in X$, $coS(x) \subset T(x)$ and $S(x) \neq \emptyset$,*
(2) *for each $y \in Y$ with $S^{-1}(y) \neq \emptyset$, there exists a nonempty open subset O_y of X such that $O_y \subset S^{-1}(y)$,*
(3) $\bigcup O_y = X$.

Then T has a continuous selection.

Now we prove the equivalence between Theorem 2.1 and Theorem 2.2;

(1) It is obvious that Theorem 2.2 implies Theorem 2.1.

(2) Now we prove that Theorem 2.1 implies Theorem 2.2. By using the O_y in condition (2) of Theorem 2.2, we define a correspondence $S_0 : X \to 2^Y$ by $S_0(x) = \{y \in Y : x \in O_y \subset S^{-1}(y)\}$. Then by conditions (2) and (3) of Theorem 2.2, $S_0(x) \neq \emptyset$ for all $x \in X$. If $y \in S_0(x)$, then $x \in O_y \subset S^{-1}(y)$ and hence $y \in S(x)$. Thus, by condition (1) of Theorem 2.2, we have for each $x \in X$, $coS_0(x) \subset coS(x) \subset T(x)$, i.e., for each $x \in X$, $coS_0(x) \subset T(x)$. If $w \in S_0^{-1}(y)$, then $y \in S_0(w)$ and thus $w \in O_y \subset S^{-1}(y)$. If $z \in O_y$, then $y \in S_0(z)$. Hence for any $z \in O_y$, $z \in S_0^{-1}(y)$. Thus $w \in O_y \subset S_0^{-1}(y)$. Therefore for each $y \in Y, S_0^{-1}(y)$ is open. Hence all conditions of Theorem 2.1 are satisfied. By Theorem 2.1, T has a continuous selection. Hence Theorem 2.1 implies Theorem 2.2.

Now we give another equivalent form of Theorem 2.1 as follows;

THEOREM 2.3. *Let X be a nonempty paracompact subset of a Hausdorff topological space and Y a nonempty convex subset of a Hausdorff topological vector space. Suppose that $T : X \to 2^Y$ is a correspondence such that*

(1) *for each $x \in X$, $T(x)$ is a nonempty convex subset of Y,*
(2) *for each $y \in Y$, $T^{-1}(y)$ contains an open subset O_y of X (O_y may be empty for some y),*
(3) $\bigcup O_y = X$.

Then T has a continuous selection.

Now we prove the equivalence between Theorem 2.1 and Theorem 2.3;

(1) We will prove that Theorem 2.3 implies Theorem 2.1. Suppose that all the asumptions of Theorem 2.1 hold. Let $O_y = S^{-1}(y)$ for any $y \in Y$. Then O_y is open for any $y \in Y$. Since $S^{-1}(y) \subset (coS)^{-1}(y) := \{x \in X : y \in coS(x)\}$ and $X = \bigcup S^{-1}(y)$, $O_y \subset (coS)^{-1}(y)$ and $X = \bigcup O_y$. By Theorem 2.3, coS has a continuous selection. Since for each $x \in X$, $coS(x) \subset T(x)$, T has a continuous selection. Hence Theorem 2.3 implies Theorem 2.1.

(2) It is obvious that Theorem 2.2 implies Theorem 2.3. Since Theorem 2.1 implies Theorem 2.2, Theorem 2.1 implies Theorem 2.3.

REMARK. Recently, Tarafdar [10] proved that if X is a compact topological space, Y is an H-space and $T : X \to 2^Y$ is a correspondence, then T has a continuous selection under the assumption: for each $x \in X$, $T(x)$ is a nonempty H-convex subset of Y, and the conditions (2) and (3) of Theorem 2.3.

EXAMPLE 2.1. Let $X = [-2, 2]$ and $Y = R$. Define correspondences $S, T : X \to 2^Y$ by

$$S(x) = \begin{cases} \{0\} & \text{if } [-2,-1] \text{ or } [1,2] \\ \{0\} \bigcup (-2x-1,\ 1] & \text{if } -1 < x < -\frac{1}{2} \\ (0,1] & \text{if } -\frac{1}{2} \leq x \leq \frac{1}{2} \\ \{0\} \bigcup (2x-1,\ 1] & \text{if } \frac{1}{2} < x < 1 \end{cases}$$

and

$$T(x) = \begin{cases} [0,1] & \text{if } |x| \leq 1 \\ \{0\} & \text{if } [-2,-1) \text{ or } (1,2], \end{cases}$$

respectively.
Then all the assumptions of Theorem 2.1 or Theorem 2.2 are satisfied, and hence T has a continuous selection. Of course, all the assumptions of Theorem 2.3 are satisfied, and hence we know that T has a continuous selection. Since $T^{-1}(y)$ is not open for each $y \in (0, 1]$, we can not apply the selection theorem of Yannelis-Prabhakar [11, Theorem 3.1] to the above example.

It is worth noticing that S does not have a continuous selection.

3. Existence of equilibria

In this section, we shall give some applications of Theorem 2.1. First we have the following fixed point theorem of Gale and Mas-Collel type [3].

THEOREM 3.1. *Let I be an (possibly infinite) index set. Let for each $i \in I$, X_i be a nonempty convex subset of a locally convex Hausdorff topological vector space E and D_i a nonempty convex compact subset of X_i. Let for each $i \in I$, $P_i : X := \prod_{i \in I} X_i \to 2^{D_i}$ be a convex(possibly empty) valued correspondence such that*

(1) *for each $y \in D_i$ with $P_i^{-1}(y) \neq \emptyset$, there exists a nonempty open subset $O_y{}^i$ of X such that $O_y{}^i \subset P_i^{-1}(y)$.*
(2) $\bigcup O_y{}^i = \bigcup \{P_i^{-1}(y) : y \in D_i, P_i^{-1}(y) \neq \emptyset\}$.

Let for each $i \in I, A_i : X \to 2^{D_i}$ be a nonempty convex-valued, closed-valued and upper semicontinuous correspondence such that for each $x \in X$ and each $i \in I$, $P_i(x) \subset A_i(x)$.If for each $i \in I$, $\{x \in X : P_i(y) \neq \emptyset\}$ is paracompact, then there exists $\bar{x} \in \prod_{i \in I} A_i(\bar{x})$ such that for each $i \in I$, either $P_i(\bar{x}) = \emptyset$ or $\bar{x}_i \in P_i(\bar{x})$, where $\pi_i(\bar{x}) = \bar{x}_i$ and π_i is the projection of X on X_i.

PROOF. For each i, let $U_i = \{x \in X : P_i(x) \neq \emptyset\}$. Since $U_i = \bigcup {O_y}^i, U_i$ is open in X. By assumptions and Theorem 2.1(or Theorem 2.2) there exists a continuous function $f_i : U_i \to D_i$ such that for each $x \in U_i, f_i(x) \in P_i(x)$. Define $\psi_i : X \to 2^{D_i}$ by

$$\psi_i(x) = \begin{cases} \{f_i(x)\} & \text{if } x \in U_i \\ A_i(x) & \text{if } x \notin U_i. \end{cases}$$

Since U_i is open, by assumptions, ψ_i is convex-valued, closed-valued and upper semicontinuous. Define $\psi : X \to 2^{\prod D_i}$ by for any $x \in X$, $\psi(x) = \prod_{i \in I} \psi_i(x)$. By Fan's lemma [2], $\prod_{i \in I} D_i$ is compact and ψ is upper semicontinuous. Since ψ is convex-valued and closed-valued, by Himmelberg's fixed point theorem [5], there exists $\bar{x} \in X$ such that $\bar{x} \in \psi(\bar{x})$. For each i, if $\bar{x} \notin U_i$, then $\bar{x}_i \in A_i(\bar{x})$ and $P_i(\bar{x}) = \emptyset$; if $\bar{x} \in U_i$, then $\bar{x}_i \in \psi_i(x) = \{f_i(\bar{x})\} \subset P_i(\bar{x}) \subset A_i(\bar{x})$. Hence there is $\bar{x} \in \prod_{i \in I} A_i(\bar{x})$ such that for each $i \in I$, either $P_i(\bar{x}) = \emptyset$ or $\bar{x}_i \in P_i(\bar{x})$.

When $E = R^n$, where R^n is the n-dimensional Euclidean space, then Theorem 3.1 reduces to the following corollary;

COROLLARY 3.1. *Let I be an (possibly infinite) index set. Let for each $i \in I$, X_i be a nonempty convex set of R^n and D_i a nonempty convex compact subset of X_i. Let for each $i \in I, P_i : X := \prod_{i \in I} X_i \to 2^{D_i}$ be a convex (possibly empty) valued correspondence such that*

(1) *for each $y \in D_i$ with ${P_i}^{-1}(y) \neq \emptyset$, there exists a nonempty open subset ${O_y}^i$ of X such that ${O_y}^i \subset {P_i}^{-1}(y)$.*
(2) $\bigcup {O_y}^i = \bigcup \{{P_i}^{-1}(y) : y \in D_i,\ {P_i}^{-1}(y) \neq \emptyset\}$.

Let for each $i \in I$, $A_i : X \to 2^{D_i}$ be a nonempty convex-valued, closed-valued and upper semicontinuous correspondence such that for each $x \in$

X and each $x \in X$ and each $i \in I$, $P_i(x) \subset A_i(x)$. Then there exists $\bar{x} \in \prod_{i\in I} A_i(\bar{x})$ such that for each $i \in I$, either $P_i(\bar{x}) = \emptyset$ or $\bar{x}_i \in P_i(\bar{x})$.

Let I be an (possibly infinite) set of agents. For each $i \in I$, let its choice set X_i be a nonempty set in a topological vector space. Let $X := \prod_{i\in I} X_i$. For each $i \in I$, let $\pi_i : X \to X_i$ be the projection of X on X_i and for each $x \in X$, let x_i denote the projection $\pi_i(x)$ of X on X_i. Let $P_i : X \to 2^{X_i}$ be an irreflexive preference correspondence, i.e., $\pi_i(x) = x_i \notin P_i(x)$ for any $x \in X$. Following [4], the collection $\Gamma = (X_i, P_i)_{i\in I}$ will be called a qualitative game. A point $\bar{x} \in X$ is said to be an equilibrium of the game Γ if $P_i(\bar{x}) = \emptyset$ for all $i \in I$.

From Theorem 3.1, we can obtain the following equilibrium existence theorem for a qualitative game in a locally convex Hausdorff topological vector space.

THEOREM 3.2. *Let $\Gamma = (X_i, \Gamma_i)_{i\in I}$ be a qualitative game such that*

(1) *X_i is a nonempty convex subset of a locally convex Hausdorff topological vector space and D_i is a nonempty convex compact subset of X_i.*

(2) *$P_i : X := \prod_{i\in I} X_i \to 2^{D_i}$ is a convex (possibly empty) valued preference correspondence such that*
   - *(i) $x_i \notin P_i(x)$ for any $x \in X$.*
   - *(ii) for any $y \in D_i$ with $P_i^{-1}(y) \neq \emptyset$, there exists a nonempty open subset $O_y{}^i$ of X such that $O_y{}^i \subset P_i^{-1}(y)$.*
   - *(iii) $\bigcup O_y{}^i = \bigcup\{P_i^{-1}(y) : y \in D_i, P_i^{-1}(y) \neq \emptyset\}$*

(3) *the set $\{x \in X : P_i(x) \neq \emptyset\}$ is paracompact in X.*

Then Γ has an equilibrium point.

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