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ON THE GENERAL VOLODIN SPACE

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ABSTRACT. We first generalize the Volodin space which Volodin constructed in order to define a new algebraic K-theory. We investigate the toplogical(homotopy) properties of the general Volodin space. We also provide a theorem which seems to be useful in pure homotopy theory. We prove that $V(\underset{\alpha}{*}\, G_\alpha, \{G_\alpha\})$ is simply connected.

1. Intrduction

Let G be a group. For a collection $\{G_\alpha\}$ of subgroups of G, the general Volodin space $X(\{G_\alpha\})$ is defined to be $\bigcup_\alpha BG_\alpha$. It is called so in this paper because Volodin used this space in defining a new algebraic K-theory. Let $G = GL_n(R)$ for a ring R. For a partial ordering α on $\{1, \cdots, n\}$ the subgroup of triangular matrices is defined by

$$T_n^\alpha(R) = \{M \in GL_n(R) \mid M_{ij} = \delta_{ij} \text{ unless } i \overset{\alpha}{<} j\}$$

In this case the Volodin space $X(\{T_n^\alpha(R)\}) = W(GL_n(R), \{T_n^\alpha(R)\})/G$ $= \bigcup_\alpha BT_n^\alpha(R)$ is simply denoted by $X_n(R)$. The space $X(R) = \varinjlim X_n(R)$ is called the Volodin model for algebraic K-theory. The space $V(GL_n(R), \{T_n^\sigma(R)\})$ is used to define Volodin K-theory as follows (cf. [5], [6]) :

$$K_i^V(R) := \pi_{i-1}(\varinjlim_n V(GL_n(R), \{T_n^\sigma(R)\}) \qquad (i \geq 3)$$

In the case of hermitian K-theory we also have the Volodin-type K-theory. In this paper we deal with some homotopy theoretic properties

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of the general Volodin space. In section 2 we review some definitions and facts about simplicial sets which are not easily collected in the existing literature. In section 3 we prove that the map $W(G, \{G_\alpha\}) \to V(G, \{G_\alpha\})$ is a homotopy equivalence. $W(G, \{G_\alpha\})$ and $V(G, \{G_\alpha\})$ have their own advantages depending on situations. In Theorem 3.5 we look closer into the space $V(G, \{G_\alpha\})$.

2. Preliminaries

DEFINITION 2.1. Let Δ be a category whose objects are $[n] = \{0, 1, \cdots, n\}$, $n \geq 0$, and a morphism $[m] \to [n]$ is a nondecreasing function. Δ is called the simplicial category. A simplicial object is a contravariant functor from Δ to a category $\mathcal{C}$ which is usually a category of sets, topological spaces, abelian groups or rings etc. We denote a simplicial object $X : \Delta^{op} \to \mathcal{C}$ by X_*. X_n denotes $X([n])$, which is called the set of n-simplices. $g^* : X_n \to X_m$ denotes $X(g : [m] \to [n])$. A simplicial map between two simplicial objects $f_* : X_* \to Y_*$ is a natural transformation from X_* to Y_*.

There is an alternative definition of simplicial object. A simplicial object is a collection of objects $\{X_n\}_{n \geq 0}$ together with functions $d_i : X_n \to X_{n-1}$ $(0 \leq i \leq n)$ and $s_i : X_n \to X_{n+1}$ $(0 \leq i \leq n)$ satisfying the following relations

$$\begin{aligned}
d_i d_j &= d_j d_i \qquad &&\text{if } i < j \\
s_i s_j &= s_j s_{i-1} \qquad &&\text{if } i > j \\
d_i s_j &= \begin{cases} s_{j-1} d_i & \text{if } i < j \\ identity & \text{if } i = j, j+1 \\ s_j d_{i-1} & \text{if } i > j+1. \end{cases}
\end{aligned}$$

We call d_i's face maps and s_i's degeneracy maps. A simplicial map $f : X_* \to Y_*$ is a collection of maps $f_n : X_n \to Y_n$ $(n \geq 0)$ which commute with face and degeneracy maps. For a simplicial object X, a simplex $x \in X$ is called degenerate if $x = s_i(x')$ for some $x' \in X$ and i. Otherwise it is called nondegenerate. Let Gps be a category of groups. A simplicial group means a simplicial object $\Delta \to Gps$. Every simplicial object can be regarded as a topological space (actually a CW-complex) via its geometric realization.

DEFINITION 2.2. Let X_* be a simplicial object. Let Δ^n be the n-simplex. The geometric realization $|X_*|$ of X is defined to be the quotient of $\coprod_{n\geq 0} X_n \times \Delta^n$ by the following relations: Let $x \in X_n$ and $\lambda : [m] \to [n]$ in Δ. Then $\Big(\lambda^* x, (t_0, \cdots, t_m)\Big) \approx \Big(x, \lambda_*(t_0, \cdots, t_m)\Big)$. We may think of $[m]$ as vertices of Δ^m, so $\lambda_* : \Delta^m \to \Delta^n$ is a linear extension of λ on barycentric coordinates. $|X_*|$ is clearly a CW-complex.

EXAMPLE 2.3. (Classifying space of a category) let $\mathcal{C}$ be a small category. Then we can form a simplicial set $B_*\mathcal{C}$, which is called bar construction (or nerve) of $\mathcal{C}$: Let $B_0\mathcal{C} = obj(\mathcal{C})$. For $n \geq 1$, n-simplices $B_n\mathcal{C}$ is a set of all possible chains of morphisms of the form

$$A_0 \xrightarrow{\alpha_1} A_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} A_n, \qquad A_i \in obj\mathcal{C}, \alpha_j \in mor\mathcal{C}$$

The i-th face map d_i deletes the i-th object and composes maps if necessary. The i-th degeneracy map s_i replaces A_i by $A_i \xrightarrow{id} A_i$. The classifying space $B\mathcal{C}$ is defined by $B\mathcal{C} = |B_*\mathcal{C}|$. $B\mathcal{C}$ may also be regarded as the space of commutative diagrams in $\mathcal{C}$. In particular, for a group G, we may regard G as a category with a single object $*$ and morphisms $* \xrightarrow{g} *$ for all $g \in G$. BG turns out to be the Eilenberg-MacLane space $K(G, 1)$.

For more details of simplicial set and classifying spaces, the readers refer to [2], [3] and [4].

DEFINITION 2.4. Let $\mathcal{C}$ be a Top-like category, i.e., category of sets, simplicial sets or topological spaces etc. A monad is an endofunctor $F : \mathcal{C} \to \mathcal{C}$ together with natural transformations $\mu : FF \to F$ and $\eta : 1 \to F$ satisfying associativity and unicity ([2], p10). An object X in $\mathcal{C}$ is a called an F-algebra if there exists a map $ev : FX \to X$ compatible with μ and η. An F-functor is a functor G together with a natural transformation $\lambda : GF \to G$ compatible with μ and η. In particular, F itself is an F-functor together with $\mu : FF \to F$. Given an monad F, and F-algebra X and an F-functor G, the two-sided bar construction $B_*(G, F, X)$ is defined to be a simplicial object in $\mathcal{C}$ whose n-simplices are GF^nX. The face maps are induced by $\eta : 1 \to F$. The two-sided bar construction has the following property:

$$\text{i)}X \to B_*(F,F,X) \text{ and } B_*(F,F,X) \to X$$
$$\text{ii)}GX \to B_*(G,F,FX) \text{ and } B_*(G,F,FX) \to X$$

are inverse homotopy equivalences.

3. The Volodin space

In this section we introduce a simplicial set $W(G,\{G_\alpha\})$ and a simplicial complex $V(G,\{G_\alpha\})$ and show that they are homotopy equivalent. We introuce the Volodin space and state some important properties of the space. The Volodin space plays an important role in algebraic K-theory.

DEFINITION 3.1. Let G be a group and $\{G_\alpha\}$ be a family of its subgroups. Denote by $V(G,\{G_\alpha\})$ a simplicial complex whose vertices are elements of G and $g_0,\cdots,g_p$ $(g_i \neq g_j)$ form a p-simplex if and only if all $g_i^{-1}g_j$ lie in the same G_k for some k. Denote by $W(G,\{G_\alpha\})$ the geometric realization of the simplicial set whose p-simplices are the sequences $(g_0,\cdots,g_p)$ of elements of G such that all $g_i^{-1}g_j$ lie in the same G_α and the face and degeneracy maps are omittings and repeatings, respectively. Note that $W(G,\{G_\alpha\}) \cong \bigcup_\alpha B(G,G_\alpha,*)$, where $B(G,G_\alpha,*)$ is the two-sided bar construction. The homeomorphism is given by

$$(g_0,\cdots,g_p) \mapsto (g_0, g_0^{-1}g_1, g_1^{-1}g_2,\cdots,g_{p-1}^{-1}g_p)$$

The general Volodin space $X(\{G_\alpha\})$ is defined by $\bigcup_\alpha BG_\alpha$ which equals

$$W(G,\{G_\alpha\})/G = \bigcup_\alpha B(*,\{G_\alpha\},*).$$

The following theorem may be known to some experts in homotopy theory, but the authors provide the proof because it does not seem to be in the literature.

THEOREM 3.2. *Let W be a CW-complex and V be a simplicial complex. Let $f: W \to V$ be a cellular map. Then if $f^{-1}(\Delta)$ is contractible for every closed simplex Δ in V, then f is a homotopy equivalence.*

PROOF. Let V_i be the i-skeleton of V and let $W_i = f^{-1}(V_i)$. We first show by induction that for each n, for every finite subcomplex V_n^α of V_n, $f : W_n^\alpha = f^{-1}(V_n^\alpha) \to V_n^\alpha$ is a homotopy equivalence. For $n = 0$, $f : W_0^\alpha \to V_0^\alpha$ is clearly an equivalence for each finite subcomplex V_0^α of V_0. We assume, by induction, that $f : W_i^\alpha = f^{-1}(V_i^\alpha) \to V_i^\alpha$ is an equivalence for every finite subcomplex V_i^α of V_i for $i \leq n-1$. Let $V_n^{\alpha_0}$ be an arbitrary finite subcomplex of V_n. Then $V_n^{\alpha_0} = (V_n^{\alpha_0})^{(n-1)} \cup$ $(finite\ n\text{-}simplices)$ is homotopy equivalent to $W_n^{\alpha_0}$ by the following argument : Let K_0 be any n-diemensional subcomplex of V_n such that $f : f^{-1}(K_0) \to K_0$ be an equivalence. Let $K' = K_0 \cup_{\partial\Delta^n} \Delta^n \subset V_n$. Then we can easily see that $f^{-1}(K') \to K'$ is also a homotopy equivalence from the following pushout squares (actually cofibration squares):

$$
\begin{array}{ccccccc}
f^{-1}(\partial\Delta^n) & \longrightarrow & f^{-1}(\Delta^n) & \qquad & \partial\Delta^n & \longrightarrow & \Delta^n \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
f^{-1}(K_0) & \longrightarrow & f^{-1}(K') & & K_0 & \longrightarrow & K'
\end{array}
$$

$f^{-1}(\partial\Delta) \to \partial\Delta^n$ is an equivalence by induction hypethesis. $f^{-1}(\Delta^n) \to \Delta^n$ and $f^{-1}(K_0) \to K_0$ are also equivalences by our assumptions, so $f^{-1}(K') \to K'$ is an equivalence. (Actually we are thinking of a cube diagram). Hence every finite subcomplex V_n^α of V_n is homotopy equivalent to $W_n^\alpha = f^{-1}(V_n^\alpha)$. Now for each n we have

$$W_n = \varinjlim_\alpha W_n^\alpha \to \varinjlim_\alpha V_n^\alpha = V_n \quad \text{is a homotopy equivalence.}$$

Moreover, we have

$$W = \varinjlim W_n \to \varinjlim V_n = V \quad \text{is also a homotopy equivalence.} \qquad \square$$

Theorem 3.2 seems to have some applications in homotopy theory and simplicial topology. We here have an immediate application.

LEMMA 3.3. *The obvious map* $f : W(G, \{G_k\}) \to V(G, \{G_k\})$ *is a homotopy equivalence.*

PROOF. It suffices to show that $f^{-1}(\Delta^n)$ is contractible for each simplex $\Delta^n = (g_0, g_1, \cdots, g_n)$ in $V(G, \{G_k\})$. It is easy to see that $f^{-1}(\Delta^n)$ can be regarded as a simplicial set whose p-simplices are all sequences $(h_0, \cdots, h_p)$ of elements of the set $\{g_0, g_1, \cdots, g_n\}$. $f^{-1}(\Delta^n)$ is contractible by the following general fact: For a nonempty set X the simplicial set whose p-simplices are all sequences $(x_0, \cdots, x_p)$ of elements of X (with the same face and degeneracy maps as above) is contractible. □

Both $W(G, \{G_\alpha\})$ and $V(G, \{G_\alpha\})$ have their own advantages depending on the homotopy theoretic situations. $V(G, \{G_\alpha\})$ is more naive and sometimes more transparent. $W(G, \{G_\alpha\})$, on the other hand, is often easier to work with in dealing with abstract homotopy theory or homology theory.

We now have some interesting properties of these spaces.

Let G be a group and $\{G_\alpha\}$ be a collection of subgroups of G. Then from Seifert van-Kampen theorem we get the following

LEMMA 3.4. *$\pi_1 X(\{G_\alpha\})$ is isomorphic to the amalgamated free product $\underset{\alpha}{*} G_\alpha$ if the collection $\{G_\alpha\}$ is closed under finite intersection.*

We have some more properties of $V(G, \{G_\alpha\})$.

THEOREM 3.5. *Let H be a subgroup of G generated by $\{G_\alpha\}$. Let $T = \underset{\alpha}{*} G_\alpha$. Then we have*

(a) *$V(H, \{G_\alpha\})$ is a connected component of $V(G, \{G_\alpha\})$.*
(b) *$V(T, \{G_\alpha\})$ is a universal covering of both $X(\{G_\alpha\})$ and $V(H, \{G_\alpha\})$.*

PROOF. (a) Since every element of H is expressed as a finite product of elements in some G_α's, every vertex in $V(H, \{G_\alpha\})$ is joined with 1 by a path. On the other hand if $g \notin H$, then g cannot be joined with 1 by a path.

(b) It suffices to show that $V(T, \{G_\alpha\})$ is simply connected. Let w be a loop at 1 in $V(T, \{G_\alpha\})$. Then w is homotopic to a loop that consists of 1-simplices. So up to homotopy every path in $V(T, \{G_\alpha\})$ can be identified with a finite product of elements g_α's, where g_α is contained in some G_α. Since w is a loop at 1, w is realized as $\prod_{finite} g_\alpha = 1$, $g_\alpha \in G_\alpha$. The group T is a free group modulo some relations. Through finite steps of reduction using these relations, $\prod g_\alpha$ reduces to 1. Each reduction happens in some G_α, which means that it geometrically happens in a simplex. Hence in each reduction the homotopy type of the loop does not change. Thus this loop w is homotopic to a trivial loop. $\square$

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