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POINTWISE PERFECT FUZZY SEMI-TOPOGENOUS ORDERS

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ABSTRACT. The purpose of this paper is to introduce a new fuzzy semi-topogenous order which agrees with the fuzzy points and investigate some properties of this order and define a fuzzy proximity structure by using this order.

1. Introduction

In [3] A. K. Katsaras and C. G. Petalas introduced a fuzzy semi-topogenous order to introduce the fuzzy syntopogenous structrue and gave a formula of determining the coarsest of all perfect (biperfect) fuzzy semi-topogenous orders finer than given fuzzy semi-topogenous order. By observing that every fuzzy point need not be completely join-irreducible (a fuzzy point x_a is completely join-irreducible iff $x_a = \vee_{j \in J} x_{aj}$ implies that $x_a \leq x_{aj}$ for some $j \in J$), we obtain the following: given a fuzzy semi-topogenous order $\ll$ on a set $X, \mu \ll^p \rho$ does not in general imply $x_a \ll \rho$ for all $x_a \leq \mu$.

2. Preliminaries

A fuzzy set, in a set X, is an element of the set I^x of all functions from X to the unit interval I. We will denote fuzzy sets by lower case Greek letters such as μ, σ, ρ. For any number $a \in I$ and any $x \in X$ we designate $x_a \in I^x$ as the function defined by $x_a(y) = 0$ if $y \neq x$ and $x_a(x) = a$.

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If $(r_j)_{j\in J}$ is a set of real numbers, we will denote by $\wedge_{j\in J}\ r_j$ and $\vee_{j\in J}\ r_j$ the $\inf_{j\in J} r_j$ and $\sup_{j\in J} r_j$, respectively. For a family $\{u_i\}_{j\in J}$ of fuzzy sets in X, the fuzzy sets $\mu = \vee_{j\in J}\ \mu_j$ and $\rho = \wedge_{j\in J}\rho_j$ are defined by $\mu(x) = \vee_j\mu_j(x), \rho(x) = \wedge_j\rho_j(x)$.

If f is a function from a set X to a set Y and $\mu \in I^Y$, then $f^{-1}(\mu)$ is the fuzzy set in X defined by $f^{-1}(\mu)(x) = \mu(f(x))$. Also, for $\sigma \in I^x, f(\sigma)$ is the fuzzy set in Y defined by $f(\sigma)(y) = 0$ if $y \notin f(X)$ and $f(\sigma)(y) = \vee\{\sigma(x) : x \in f^{-1}(y)\}$ if $y \in f(X)$.

For the definitions and notations of a fuzzy semi-topogenous orders we will refer to [3].

3. Pointwise perfect fuzzy semi-topogenous orders

Let $\ll$ be a fuzzy semi-topogenous order on a set X. For $x \in X$ and $\mu \in I^x$, we will let $x_{\mu(\ll)}$ denote $\vee\{x_a : x_a \ll \mu,\ a \in I\}$.

DEFINITION 3.1. We say that a fuzzy semi-topogenous order $\ll$ on a set X is a pointwise perfect if it satisfies the following:

$f)\mu \ll \rho$ and $\sigma \leq \vee_{x\in F}\ X_{\rho(\ll)}$ for some finite subset F of X imply $\mu \vee \sigma \ll \rho$.

REMARK 3.2. 1) Let $\ll$ be a pointwise perfect fuzzy semi-topogenous order on a set X. Then $x_{\mu(\ll)} \ll \mu$ for any $\mu \in I^x$ and $x \in X$.

2) If $\{\ll_j\colon j \in J\}$ is a non-empty family of pointwise perfect fuzzy semi-topogenous orders on a set X, then the intersection $\cap_{j\in J} \ll_j$ is also pointwise perfect.

3) Every perfect fuzzy semi-topogenous order is pointwise perfect.

If $\ll$ is a fuzzy semi-topogenous order, then we can determine the coarsest of all pointwise perfect fuzzy semi-topogenous orders finer than $\ll$. This is contained in the followig theorem.

THEOREM 3.3. *For any fuzzy semi-topogenous order $\ll$ on a set X, there exists a pointwise perfect fuzzy semi-topogenous order $\ll^f$ finer than $\ll$ and coarser than all other pointwise perfect semi-gopogenous orders which are finer than $\ll$. This order $\ll^f$ is defined in the following way:*

(3.3) $\mu \ll^f \rho$ means that there exists a decomposition $\mu = \sigma \vee \nu$ such that $\sigma \ll \rho$ and $\nu \leq \vee_{x\in F}\ x_{\rho(\ll)}$ for some finite subset F of X.

PROOF. It is easy to show that $\ll^f$ is a fuzzy semi-topogenous order on X. To show that $\ll^f$ is a pointwise perfect, let $\mu \ll^f \rho$. There are fuzzy sets ρ and ν such that $\sigma \ll \rho$ and $\nu \leq \vee_{x\in F}\, x_{\rho(\ll)}$ for some finite subset F of X. If $\tau \leq \vee_{x\in G}\, x_{\rho(\ll)}$ for some finite subset G of X, then $\mu \vee \tau \leq \sigma \vee (\nu \vee \tau) \ll^f \rho$ since $G \cup F$ is a finite subset of X and $\sigma \vee \tau \leq \vee_{x\in F\cup G}\, x_{\rho(\ll)}$. It is clear that $\mu \ll \rho$ implies that $\mu \ll^f \rho$ and hence $\ll^f$ is finer than $\ll$. Finally let $\ll_0$ be a pointwise perfect semi-topogenous order on X finer than $\ll$ and suppose $\sigma \ll \rho$ and $\nu \leq \vee_{x\in F}\, x_{\rho(\ll)}$ for some finite subset F of X. Since $\ll_0$ is finer than $\ll$, we have that $\sigma \ll_0 \rho$ and $\nu \leq \vee_{x\in F}\, x_{\rho(\ll)} \leq \vee_{x\in F}\, x_{\rho(\ll_0)}$. Since $\ll_0$ is pointwise perfect, $\mu \ll_0 \rho$. This proves that $\ll^f$ is coarser than $\ll_0$. The proof is complete.

By the definition of $\ll^f$, the following propositions are easily obtained.

PROPOSITION 3.4. *Let $\ll$ be a fuzzy semi-topogenous order on a set X. Then one has the following:*
1) $\ll$ is pointwise perfect iff $\ll = \ll^f$,
2) $\ll^{ff} = \ll^f$.

The following corollaries are immediate consequences of the previous theorems.

COROLLARY 3.5. *If $\{\ll_j : j \in J\}$ is any non-void family of pointwise perfect fuzzy semi-topogenous orders, then 1) $(\cup_{j\in J} \ll_j)^f = (\cup_{j\in J} \ll_j^f)^f$, 2) $(\cup_{j\in J} \ll_j)^f$ is the coarsest of all pointwise perfect fuzzy semi-topogenous orders finer than each of the given fuzzy semi-topogenous orders $\ll_j$.*

PROPOSITION 3.6. *Let $\ll_1$ and $\ll_2$ be semi-topogenous orders on a set X such that $\ll_1$ is coarser than $\ll_2$. Then $\ll_1^f$ is coarser than $\ll_2^f$.*

PROOF. Suppose that $\mu \ll_1^f \rho$. There are fuzzy sets σ and ν such that $\sigma \ll_1 \rho$ and $\nu \leq \vee_{x\in F}\, x_{\rho(\ll_1)}$ for some finite subset F of X. Since $\ll_1$ is coarser than $\ll_2$, $\sigma \ll_2 \rho$ and $\nu \leq \vee_{x\in F}\, x_{\rho(\ll_2)}$ for some finite subset F of X and so $\mu \ll_2^f \rho$.

THEOREM 3.7. *For any fuzzy semi-topogenous order $\ll$, we have $\ll^{fq}$ is a coarser than $\ll^{qf}$.*

PROOF. Suppose $\mu \ll^{fq} \rho$, we have because of (3.1) in [3] $\mu = \vee_{i=1}^{m}\mu_i$, $\rho = \wedge_{j=1}^{n}\rho_j$ and $\mu_i \ll^{f} \rho_j$ for all $i = 1, \cdots, m$ and all $j = 1, \cdots, n$. For each i, there exists a decomposition $\mu_i = \sigma_i \vee \nu_i$ such that $\sigma_i \ll \rho_j$ and $\nu_i \leq \vee_{x \in Fi}\, x_{\rho j(\ll)}$ for some finite subset F_i of X.

Since $\mu = \vee_{i=1}^{m}(\sigma_i \vee \nu_i) = (\vee_{i=1}^{m}\sigma_i) \vee (\vee_{i=1}^{m}\nu_i)$, $\vee_{i=1}^{m}\sigma_i \ll^{q} \rho_j$, $(\vee_{i=1}^{m}\nu_i) \leq (\vee_{x \in \cup Fi}\, x_{\rho j(\ll)})$ and $\cup_1^m Fi$ is finite,

$$(3.7.1) \qquad \mu \ll^{qf} \rho_j \ (j \in G). \qquad \text{where} \qquad G = \{1, 2, \cdots, n\}$$

(3.7.1) implies that for each $j \in G$, there exists a decomposition $\mu = \sigma_j \vee \nu_i$ such that $\sigma_j \ll^{q} \rho_j$ and $\nu_j \leq (\vee_{x \in Fj}\, x_{\rho j(\ll^q)})$ for some finite subset F_j of X.

Then $\mu = \wedge_{j \in G}(\sigma_j \vee \nu_j) = \vee_{\phi \neq A \subseteq G}\ ((\wedge_{j \in A}\sigma_j) \wedge (\wedge_{j \in G-A}\nu_j)) \vee (\wedge_{j \in G}\nu_j)$.

Since $\vee_{\phi \neq A \subseteq G}((\wedge_{j \in A}\sigma_j) \wedge (\wedge_{j \in G-A}\nu_j)) \ll^{q} \wedge_{j \in G}\rho_i = \rho$, $\wedge_{j \in G}\, \nu_j \leq \vee_{x \in \cup Fj} x_{\rho(\ll^q)}$ and $\cup_{j \in G}F_j$ is finite, $\mu \ll^{qf} \rho$, which completes the proof.

By the above theorem and (3.2) in [3], the following corollaries are obtained easily.

COROLLARY 3.8. *1) If $\ll$ is a fuzzy topogenous order on a set X, then $\ll^{f}$ is also a fuzzy topogenous order.*

2) For any fuzzy semi-topogenous order $\ll$, the pointwise perfect fuzzy topogenous order $\ll^{qf}$ is the coarsest of all pointwise perfect fuzzy topogenous orders finer than $\ll$.

Let f be a function from a set X to a set Y and let $\ll$ be a fuzzy semi-topogenous order on Y. Define a binary relation $\ll_0$ on I^X by

$$\mu \ll_0 \rho \ \text{ iff } \ f(\mu) \ll 1 - f(1 - \rho).$$

Then we say that $\ll_0$ is the inverse image of $\ll$ by the mapping f and we denote it by $f^{-1}(\ll)$ ([3]).

THEOREM 3.9. *If f is a function from a set X to a set Y and $\ll$ is a fuzzy semi-topogenous order on Y, then $f^{-1}(\ll)^f$ is coarser than $f^{-1}(\ll^f)$.*

PROOF. Suppose that $\mu f^{-1}(\ll)\rho$. Then there is a decomposition $\mu = \sigma \vee \nu$ such that $\sigma f^{-1}(\ll)\rho$ and $\nu \leq \vee_{x\in F} X_{\rho(f^{-1}(\ll))}$ for some finite subset F of X.

From the definition of $f^{-1}(\ll)$,

$$f(\sigma) \ll 1 - f(1-\rho). \tag{3.9.1}$$

On the other hand,

$$f(\nu) \leq \vee_{x\in F}(\vee\{f(x_p) = f(x)_p : f(x_p) \ll 1 - f(1-\rho)\})$$
$$\leq \vee_{x\in F}\, f(x)_{((1-f(1-\rho))(\ll))}.$$

Put $G = f(F)$. Then we have

$$f(\nu) \leq \vee_{y\in G}\, y_{1-f(1-\rho)}. \tag{3.9.2}$$

Thus, by (3.9.1) and (3.9.2), $f(\mu) = f(\sigma \vee \nu) = f(\sigma) \vee f(\nu) \ll^f 1 - f(1-\rho)$, and so $\mu f^{-1}(\ll^f)\rho$. The proof is complete.

It is immediate from the above proposition and Theorem 3.3.

COROLLARY 3.10. *If f is a function from a set X to a set Y and $\ll$ is a pointwise perfect fuzzy semi-topogenous order on Y, then $f^{-1}(\ll)$ is also pointwise perfect.*

PROPOSITION 3.11. *If f is a function from a set X to a set Y and for each $y \in Y$, $f^{-1}(\{y\})$ is finite, then for any fuzzy semi-topogenous order $\ll_0$ on Y,*

$$f^{-1}(\ll_0^f) = f^{-1}(\ll_0)^f.$$

PROOF. Put $\ll = f^{-1}(\ll_0)$ and $A(y) = f^{-1}(\{y\})(y \in Y)$. By Theorem 3.9 we need only show that $f^{-1}(\ll_0^f)$ is coarser than $\ll^f$. Suppose that $\mu f^{-1}(\ll_0^f)\rho$. Then there exists a decompostion $f(\mu) = \sigma \vee \nu$ such that $\sigma \ll_0 1 - f(1-\rho)$ and $\nu \leq \vee_{y\in F}\, y_{((1-f(1-\rho))(\ll))}$ for some finite subset F of Y. From the definition of $\ll$,

$$f^{-1}(\sigma) \ll \rho. \tag{3.11.1}$$

Since $f^{-1}(y_p) = \vee_{x\in A(y)} x_p$,

$$f^{-1}(\nu) \leq \vee_{y\in F} f^{-1}(y_{(1-f(1-\rho))(\ll)}) \\ = \vee_{y\in F} (\vee\{f^{-1}(y_p) : y_p \ll_0 1 - f(1-\rho)\} \\ = \vee_{y\in F} (\vee\{\vee_{x\in A(y)} x_p : y_p \ll_0 1 - f(1-\rho)\}.$$

Since for each $x \in A(y)$, $x_p \leq x_{\rho(\ll)}$,

$$f^{-1}(\nu) \leq \vee_{x\in\cup\{A(y):y\in F\}} x_{p(\ll)}. \tag{3.11.2}$$

Since for each $y \in Y$, $A(y)$ is finite, $\cup\{A(y) : y \in Y\}$ is finite. Thus, by (3.11.1) and (3.11.2), $\mu \leq f^{-1}(f(\mu)) = f^{-1}(\sigma\vee\nu) = f^{-1}(\sigma)\vee f^{-1}(\nu) \ll^f \rho$, and so $\mu \ll^f \rho$. The proof is complete.

REMARK 3.12. Let X be a set and for each natural number $n \geq 3$ and any $x \in X$, put $\mu_n^x = x_{(1/2-1/n)}$. Define a binary relation order $\ll_x$ on I^x as follows:

$\mu \ll_x \rho$ iff $\mu = 0$, $\rho = 1$, or $\mu \leq \mu_n^x$ for some n and $x_{1/2} \leq \rho$. It is easy to show that $\ll_x$ is a fuzzy semi-topogenous order on X and that $x_{1/2} \ll_x^p x_{1/2}$ holds b ut not $x_{1/2} \ll_x x_{1/2}$.

THEOREM 3.13. *If $\ll$ is a pointwise perfect fuzzy semi-topogenous order on X, the perfect fuzzy semi-topogenous order $\ll^p$ can be defined as follwos;*

(3.31) *$\mu \ll^p \rho$ means that $x_a \ll \rho$ for every fuzzy point $x_a \leq \mu$.*

PROOF. Suppose $x_a \leq \mu$. Since $\mu \ll^p \rho$, there is a decomposition $\mu = \vee_{j\in J} \mu_j$ such that $\mu_j \ll \rho$ $(j \in J)$ and hence $x_a \leq \vee_{j\in J} \mu_j(x) \leq x_{\rho(\ll)}$. Since $\ll$ is a pointwise perfect fuzzy semi-topogenous order on X, $x_a \leq x_{\rho(\ll)} \ll \rho$ which implies $x_a \ll \rho$. Conversely, if we have $x_a \ll \rho$ for any $x_a \leq \mu$, we can put

$$\mu = \vee\{x_a : x_a \leq \mu\}$$

which shows $\mu \ll^p \rho$.

PROPOSITION 3.14. *If both $\ll_0$ and $\ll_0^c$ are pointwise perfect fuzzy semi-topogenous orders on a set X, then $\ll_0^{cpc}$ is also pointwise perfect.*

PROOF. Suppose $\mu \ll \rho$ and $\nu \leq \vee_{x\in F}\, x_{\rho(\ll)}$ for some finite subset F of X, where $\ll = \ll_0^{cpc}$. Since $\ll^c$ is pointwise perfect,

$$\begin{aligned} x_{\rho(\ll)} &= \vee \{x_p : x_p \ll \rho\} \\ &= \vee \{x_p : 1-\rho \ll_0^{cp} -1 - x_p\} \\ &= \vee \{x_p : z_a \ll_0^c 1 - x_p, \text{ for all } \ z_a \leq 1-\rho\} \\ &= \vee \{x_p : x_p \ll_0 1 - z_a, \text{ for all } \ z_a \leq 1-\rho\} \qquad (x \in F). \end{aligned}$$

Thus $\nu \leq 1 - z_a$ for all $z_a \leq 1-\rho$. Since $\mu \ll_0 1 - z_a$ $(z_a \leq 1-\rho)$ and $\ll_0$ is pointwise perfect, $\mu \vee \nu \ll_0 1 - z_a$ for all $z_a \leq 1-\rho$ and so $\mu \vee \nu \ll \rho$. This completes the proof.

PROPOSITION 3.15. *For any fuzzy semi-topogenous order $\ll$, $\ll^b = \ll^{cpcp}$.*

PROOF. $\mu \ll^{cpcp} \rho$ means that there exists a decomposition

$$\mu = \vee_{j\in I}\, \mu_i \tag{3.15.1}$$

such that $\mu_i \ll^{cpc} \rho$ $(i \in I)$. This formula is equivalent to $1-\rho \ll^{cp} 1 - \mu_i$ $(i \in I)$, and to the existence of a decomposition

$$1-\rho = \vee\{\sigma_{ij} : j \in J_i\} \tag{3.15.2}$$

such that $\sigma_{ij} \ll^c 1 - \mu_i$ $(i \in I,\ j \in J_i)$ or $\mu_i \ll 1 - \sigma_{ij}$ $(i \in I,\ j \in J_j)$.

Hence from (3.15.1) and (3.15.2), which imply

$$\rho = \wedge\{1-\sigma_{ij} : j \in J_i\} = \wedge\{\wedge\{1-\sigma_{ij} : j \in J_i\} : i \in I\}$$

the formula $\mu \ll^b \rho$ holds.

Conversely, $\mu \ll^b \rho$ implies $\mu = \vee_{i\in I}\, \mu_i$, $\rho = \wedge_{j\in J}\, \rho_j$, $\mu_i \ll \rho_j$, $\mu_i \ll \rho_j$ $(i \in I,\ j \in J)$ hence $1-\rho_j \ll^c 1-\mu_i$, $1-\rho = \vee_{j\in J}(1-\rho_j) \ll^{cp} 1-\mu_i$, $\mu_j \ll^{cpc} \rho$, and finally $\mu \ll^{cpcp} \rho$.

THEOREM 3.16. *If both $\ll$ and $\ll^c$ are pointwise perfect fuzzy semi-topogenous orders on a set X, the biperfect fuzzy semi-topogenous order $\ll^b$ can be defined as follows:*
(3.16)
$\mu \ll^b \rho$ *means that* $x_a \ll 1 - y_b$ *for an arbitrary* $x_a \leq \mu,\ \ y_b \leq 1 - \rho$.

PROOF. By virtue of Proposition 3.15, $\mu \ll^b \rho$ is equivalent to $\mu \ll^{cpcp} \rho$, and this formula holds on accout of Proposition 3.14, if and only if, $x_a \ll^{cpc} \rho$ for an arbitrary $x_a \leq \mu$, or, which is the same thing $1 - \rho \ll^{cp} 1 - x_a$ for $x_a \leq \mu$. Applying once again (3.13), it can be stated that this condition is equivalent to

$$y_b \ll^c 1 - x_a, \quad \text{or} \quad x_a \ll 1 - y_b \quad \text{for} \quad x_a \leq \mu,\ \ y_b \leq 1 - \rho,$$

in other words exactly equivalent to (3.16).

The following propositions are immediately consequences of the previous propositions.

PROPOSITION 3.17. *If both $\ll$ and $\ll^c$ are pointwise perfect fuzzy semi-topogenous orders, then $\ll^{qb} = \ll^{bq} = \ll^b$ and $\ll^{pb} = \ll^{bp} = \ll^b$.*

REMARK 3.18. It follows from example in Remark 3.12 that if $a = f$ or p and $\ll$ is any fuzzy semi-topogenous order on a set X then in general, $\ll^{ac} \neq \ll^{ca}$. In fact, $1 - x_{1/2} \ll^{ac} 1 - x_{1/2}$ holds but not $1 - x_{1/2} \ll^{ca} 1 - x_{1/2}$. But there still exists an important particular case where the symmetry of a pointwise perfect fuzzy topogenous order $\ll$ implies the symmetry of $\ll^p$. This is included in the following proposition:

PROPOSITION 3.19. *If $\{\ll_j : j \in J\}$ is a finite sequence of arbitrary pointwise perfect fuzzy semi-topogenous orders on a set X, then $(\cup_1^n \ll_j)^{qp}$ is coarser than $(\cup_1^n \ll_j^{pc})^{qpc}$.*

PROOF. Put

$$(3.18.1) \qquad \ll = \cup_1^n \ll_j, \quad \ll_0 = \cup_1^n \ll_j^{pc}.$$

$\mu \ll^{qp} \rho$ implies in view of (3.13) $x_a \ll^q \rho$ for $x_a \leq \mu$, thus

$$x_a \ll^q 1 - y_b \quad \text{for} \quad x_a \leq \mu,\ \ y_b \leq 1 - \rho.$$

Hence it follows form ((3.1) in [3])

$$x_a \ll 1 - y_b \text{ for } x_a \leq \mu, \quad y_b \leq 1 - \rho. \tag{3.18.2}$$

Denote by $\mu_{yi} \vee \{x_a : x_a \ll_i 1 - y_b \text{ and } x_a \leq \mu\}$. According to (3.18.1) and 3.18.2,

$$\mu = \vee_1^n \mu_{yi} \text{ for } y_b \leq 1 - \rho.$$

On the other hand, according to (3.13) for any $y_b \leq 1-\rho$, $\mu_{yi} \ll_j^p 1-y_b$, thus $y_b \ll_i^{pc} 1 - \mu_{yi}$, and hence $y_b \ll_0 1 - \mu_{yi}$, and so it follows that according to (3.18.1)

$$y_b \ll_0^q \wedge_1^n (1 - \mu_{yi}) = 1 - \mu \qquad (y_b \leq 1 - \rho).$$

Apply (3.13) once again,

$$1 - \rho \ll_0^{qp} 1 - \mu,$$

that is, $\mu \ll_0^{qpc} \rho$, which completes the proof

By the above proposition, the following corollary is easily obtained.

COROLLARY 3.20. *If $\{\ll_j : j \in J\}$ is a finite sequence of arbitrary perfect symmetrical fuzzy semi-topogenous orders on a set X, then $(\cup_1^n \ll_j)^{qp}$ is a perfect, symmetrical topogenous structure.*

4. Correspondence between pointwise perfect fuzzy proximities and symmetrical pointwise perfect fuzzy topogenous structures

DEFINITION 4.1. If a fuzzy syntopogenous structure S on a set X consists of a single pointwise perfect fuzzy topogenous order, then S is called a pointwise perfect fuzzy topogenous structure.

If $\{\ll\}$ is a symmetrical pointwise perfect topogenous structure on a set X, we may introduce a new binary relation δ on I^x by setting $\mu \ \delta \ \rho$ iff not $\mu \ll 1 - \rho$. Then δ, which is called a pointwise perfect fuzzy proximity, satisfies the following axiom:

1) $\mu \ \delta \ \rho$ implies $\rho \ \delta \ \mu$

2) $(\mu \vee \rho)\delta\ \sigma$ iff $\mu\ \delta\ \sigma$ or $\rho\ \delta\ \sigma$,
3) $\mu\ \delta\ \rho$ implies that $\mu \neq 0$ and $\rho \neq 0$,
4) $\mu\ \delta^*\ \rho$ (ρ^* is the negation of δ) implies that $\mu \leq 1 - \rho$.
5) $\mu\ \delta^*\ \rho$ implies the existence of a $\sigma \in I^x$ such that $\mu\ \delta^*\ \sigma$ and $(1-\sigma)\ \delta^*\ \rho$.
6) $\mu\ \delta^*\ 1 - \rho$ and $\nu \leq \vee_{x \in F}\ x_{\rho(\delta)}$ for some finite subset F of X imply $\mu \vee \nu\ \delta^*\ 1 - \rho$, where $x_{\rho(\delta)}$ denotes $\vee\{x_a : x_a\ \delta^*\ 1 - \rho\}$.

The mapping $\ll \mapsto \delta_{\ll}$ is clearly one-to-one. Conversely, if δ satisfying 1) - 6) is given and we define $\mu \ll \rho$ iff $\mu\ \delta^*\ 1 - \rho$, $\{\ll\}$ is a pointwise perfect symmetrical fuzzy topogenous structure on X with $\delta = \delta_{\ll}$. Thus we have:

THEOREM 4.2. *The mapping $\ll \mapsto \delta_{\ll}$, from the set of all pointwise perfect symmetrical fuzzy topogenous structures on a set X to the set of all pointwise perfect fuzzy proximities on X, is one-to-one and onto.*

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