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THE STRUCTURE CONFORMAL VECTOR FIELDS ON A SASAKIAN MANIFOLD II

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ABSTRACT. The concept of the structure conformal vector field C on a Sasakian manifold M is defined. The existence of such a C on M is determined by an exterior differential system in involution. In this case M is a foliate manifold and the vector field C enjoys the property to be exterior concurrent. This allows to prove some interesting properties of the Ricci tensor and Obata's theorem concerning isometries to a sphere. Different properties of the conformal Lie algebra induced by C are also discussed.

0. Introduction

Let $M(\Phi, \eta, \xi, g)$ be a (2m+1)-dimensional Sasakian manifold with *soldering form* $dp \in \Gamma Hom(\Lambda^q TM, TM)$ (dp: canonical vector-valued 1-form) where Φ, η, ξ and g are the (1,1)-tensor field, the structure 1-form, the structure vector field and the metric tensor of M, respectively. Since one may write $\nabla\xi = \Phi dp$, we give the following definition : Any vector field C such that

$$\nabla C = \rho dp + \lambda \nabla \xi \ ; \qquad \rho, \lambda \in C^\infty M, \tag{0.1}$$

is defined as a *conformal vector field* ((0.1) implies $\mathcal{L}_C g = 2\rho g$)

In Section 2, it is proved that the existence of C on $M(\Phi, \eta, \xi, g)$ is determined by an exterior differential system *in involution* (in the sence of É. Cartan [4]), and that any M which carries a vector field C, is foliated by *autoparallel* three-dimensional submanifold of scalar curvature +1, tangent to $C, \Phi C$ and ξ. Besides such a Sasakian manifold

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possesses the remarkable property to be isometric to a unit sphere in a (2m+2)-dimensional Euclidean space [8].

Furthermore, any C is an *exterior concurrent* vector field (see [11], [9]) and of *conformal weight* $\frac{2m+1}{m}$ [7].

Let $\mathcal{D}_C$ be the autoparallel distribution tangent to $C, \Phi C$ and ξ. In Section 3, we consider three conformal vector fields $C_\nu \in \mathcal{D}_C, \nu = 1,2,3$ generated by C by means of the Lie bracket and agree to call them the *associated conformal vector fields of* C. If ρ and ρ_ν are the conformal scalars corresponding to C and C_ν respectively, then it is proved that they define a 4-dimensional *eigenspace* $\varepsilon^4(M)$ of the Laplacian Δ of the eigenvalue $2m+1$.

If r denotes the scalar curvature of M, then we may write

$$\mathcal{L}_C r = 2(2m(2m+1) - r)\rho.$$

In this case when M is compact, some integral formulas of Watanabe type (see [13]) are obtained.

More generally, consider a $K-$ *contact* manifold $M(\Phi, \eta, \xi, g)$, i.e. a contact metric manifold whose structure vector ξ is a Killing vector field [2].

We give the following definition : Any vector field X such that

$$\mathcal{L}_X \Omega = h\Omega + \gamma \wedge \eta \tag{0.2}$$

where $\Omega = \frac{1}{2} d\eta$, $h \in C^\infty M$, $\gamma \in \Lambda^1 M$, is called an *infinitesimal quasi-conformal contact transformation of* Ω. In particular, if X is a conformal vector field, then following [5], we will call it a *quasi-biconformal* vector field.

In Section 4, we discuss some infinitesimal transformations defined by C and its associated vector field C_ν. We prove that if R denotes the curvature tensor, then the following formula holds:

$$(\mathcal{L}_C R)(Z, Z', Z'') = 2\rho(Z \wedge Z')Z'', \qquad Z, Z', Z'' \in \mathcal{X}M$$

and the quality of C to be exterior concurrent (abbreviation : E.C.) implies

$$\mathcal{L}_C g(C, Z) = 2\rho g(C, Z).$$

This shows that C defines an infinitesimal conformal transformation of all the functions $g(C, Z)$.

We also prove that C and C_ν are quasi-biconformal vector fields with respect to the pairing (Ω, η), and that Ω is a *relatively integral invariant* of C_ν (see [1]).

1. Preliminaries

Let (M, g) be an orientable C^∞-Riemannian manifold and let ∇ be the covariant differential operator defined by the metric tensor g.

Let $\Gamma(TM)$ be the set of sections of the tangent bundle TM and $\flat : TM \to T^*M$ be the *musical isomorphism* [10] defined by g.

If, following [10], we denote by

$$A^q(M, TM) = \Gamma \mathrm{Hom}(\Lambda^q TM, TM)$$

the set of vector-valued q-forms, q < dim M, then

$$d^\nabla : A^q(M, TM) \to A^{q+1}(M, TM)$$

means the *exterior covariant derivative* operator with respect to ∇. It should be noticed that generally $d^{\nabla^2} = d^\nabla \circ d^\nabla \neq 0$, unlike d^2.

If $dp \in A^1(M, TM)$ denotes the soldering form of M, any vector field X such that

$$d^\nabla(\nabla X) = \nabla^2 X = \pi \wedge dp \in A^2(M, TM), \tag{1.1}$$

is defined as *exterior concurrent* (abbreviation :E.C.)(see [11],[9]).

It has been proved [9] that π is necessarily given by

$$\pi = \nu \flat(X); \qquad \nu \neq 0, \tag{1.2}$$

where $\nu \in C^\infty M$ is the conformal scalar associated with X.

If $\mathcal{R}$ denotes the *Ricci tensor* of ∇, it follows from (1.1) and (1.2) that

$$\mathcal{R}(X, Z) = -(n-1)\nu g(X, Z) \Rightarrow \nu = -\frac{1}{n-1}\mathrm{Ric}X, \tag{1.3}$$

where $Z \in \mathcal{X}M$ and dim M=n.

Let $\mathcal{T} \in \mathcal{X}M$ be any *conformal vector field* on M (or conformal Killing vector field), that is

$$(1.4) \qquad \mathcal{L}_{\mathcal{T}} g = 2\rho g \Leftrightarrow < \nabla_Z \mathcal{T}, Z' > + < \nabla_{Z'} \mathcal{T}, Z > = 2\rho < Z, Z' >$$

where $\rho \in C^\infty M; Z, Z' \in \mathcal{X}M$ and

$$(1.5) \qquad \rho = \frac{\mathrm{div}\mathcal{T}}{n}$$

If R and r denotes the curvature tensor and the scalar curvature of M respectively, we recall the following basic formulas (see [3], [13]):

$$(1.6) \qquad \mathcal{L}_{\mathcal{T}} \flat(Z) = 2\rho\flat(Z) + \flat[\mathcal{T}, Z], \qquad Z \in \mathcal{X}M,$$

(1.7)

$$\begin{aligned}(\mathcal{L}_{\mathcal{T}} R)(Z, Z', Z'') = &-(\mathrm{Hess}_\nabla \rho)(Z', Z'')Z + (\mathrm{Hess}_\nabla \rho)(Z, Z'')Z' \\ &- g(Z', Z'')(\nabla \mathrm{grad}\rho)Z + g(Z, Z'')(\nabla \mathrm{grad}\rho)Z',\end{aligned}$$

where $(\mathrm{Hess}_\nabla \rho)$ is a covariant and symmetric 2-tensor defined by $(\mathrm{Hess}_\nabla \rho$ $(Z, Z') = g(Z, H_\rho Z'); H_\rho Z' = \nabla_{Z'} \mathrm{grad}\rho)$; see also [3],

$$(1.8) \qquad \mathcal{L}_{\mathcal{T}} \mathcal{R}(Z, Z') = (\Delta\rho) g(Z, Z') - (n-2)(\mathrm{Hess}_\nabla \rho)(Z, Z')$$

and

$$(1.9) \qquad \mathcal{L}_{\mathcal{T}} r = (n-1)\Delta\rho - r\rho,$$

where $Z, Z', Z'' \in \mathcal{X}M$ and [,] is the Lie bracket.

If $\mathcal{T}_1$ and $\mathcal{T}_2$ are conformal vector fields, then $[\mathcal{T}_1, \mathcal{T}_2]$ is a conformal vector field with (see [3])

$$(1.10) \qquad \rho_{\mathcal{T}_1, \mathcal{T}_2} = [\mathcal{L}_{\mathcal{T}_1}, \rho_{\mathcal{T}_2}] + [\rho_{\mathcal{T}_1}, \mathcal{L}_{\mathcal{T}_2}].$$

We also recall (see [13]) that if M is a compact manifold and its Ricci curvature is negative definite, then a conformal vector field other than zero vector field does not exist.

Any vector field X such that

$$\mathcal{L}_X \flat(X) = c(\mathrm{div}X)\flat(X); \qquad c = \mathrm{const} \tag{1.11}$$

is defined as a *self conformal* vector field.

We notice that in general (even if X is not a conformal vector field) if $\dim M = n$, then cn is called the *conformal weight of* $\flat(X)$ (cf. [7]).

In addition, if M is a compact manifold, then the following integral formula holds (see [13]):

$$\int_M (\mathcal{R}(\mathcal{T}, \mathcal{T} - |\nabla \mathcal{T}|^2 - \frac{n-2}{n}(\mathrm{div}\mathcal{T})^2) = 0. \tag{1.12}$$

We also recall the following theorm of M. Obata [8] (see also [3]): In order that a gradient vecter field grad f be an *infinitesimal concircular transformation* on an n-dimensional manifold M, it is necessary and sufficient that

$$< \nabla_Z \,\mathrm{grad} f, Z' >= \nu < Z, Z' >, \qquad Z, Z' \in M, \tag{1.13}$$

where ν is a non-vanishing scalar. If $\nu = -c^2 f$, then M is isometric to a sphere S^n of radius $\frac{1}{c}$ in an $(n+1)$-dimensional Euclidean space.

2. A structure conformal vector field on a Sasakian manifold

Let $M(\Phi, \eta, \xi, g)$ be a $(2m+1)$-dimensional contact metric manifold. In such a manifold the structure tensors Φ, η and ξ satisfy (see [15]) the equations :

$$\begin{cases} \Phi\xi = 0, \qquad \eta(\xi) = 1, \\ \Phi^2 = -I + \eta \otimes \xi, \qquad \eta(Z) = g(\xi, Z), \\ g(\Phi Z, \Phi Z') = g(Z, Z') - \eta(Z)\eta(Z'), \\ g(\Phi Z, Z') = \frac{1}{2} d\eta(Z, Z'), \qquad Z, Z' \in \mathcal{X}M. \end{cases} \tag{2.1}$$

The Φ -Lie derivative is defined by

$$(\nabla\Phi)Z = \nabla\Phi Z - \Phi\nabla Z, \tag{2.2}$$

and it has been shown in [2] that ξ is a Killing vector fields if and only if $\mathcal{L}_\xi \Phi$ vanishes. In this case M is called a K-*contact manifold.* A K-contact manifold for which one has

$$(\nabla_Z \Phi)Z' = -g(Z, Z')\xi + \eta(Z')Z, \tag{2.3}$$

is called a *Sasakian manifold.*

If M is a Sasakian manifold, then ξ is always E.C. and

$$\nabla^2 \xi = -\eta \wedge dp \Rightarrow \mathcal{R}(\xi, Z) = 2mg(\xi, Z) \tag{2.4}$$

(see [9]). Moreover, any E.C.vector field X satisfies

$$\nabla^2 X = -\flat(X) \wedge dp, \tag{2.5}$$

and the property of the exterior concurrency is invariant under the action of Φ (i.e. $\nabla^2 \Phi X = -\flat(\Phi X) \wedge dp$).

In the following we shall that

$$d\eta = 2\Omega \tag{2.6}$$

and agree to call Ω the *fundamental presymplectic form* of M (abbreviation f.p.f.)

In the more general case when M is a K-contact manifold, we introduce the following two definitions

a) A vector field C on M such that

$$\nabla C = \rho dp + \lambda \nabla \xi\,; \qquad \rho, \lambda \in C^\infty M, \tag{2.7}$$

is defined as a *structure conformal vector field.* Effectively, since ξ is a Killing vector field, it is easy to see that the equation (2.7) satisfies the conformal equation, that is (see (1.4)):

$$\mathcal{L}_C g = 2\rho g \Leftrightarrow < \nabla_Z C, Z' > + < \nabla_{Z'} C, Z > = 2\rho < Z, Z' >, \tag{2.8}$$

where $Z, Z' \in \mathcal{X}M$, and this implies (see (1.5))

$$\text{div} C = (2m + 1)\rho$$

b) Since we have set $d\eta = 2\Omega$, any vector field X such that

$$\mathcal{L}_X \Omega = \Phi\Omega + \gamma \wedge \eta; \qquad \gamma \in \Lambda^1 M, \Phi \in C^\infty M, \tag{2.10}$$

is called an *infinitesimal quasi-conformal contact transformation of* Ω (abbreviation : i.q.c.c.t)

Denote by $\mu : TM \to T^*M$, $X \to i_X\Omega$ the bundle isomorphism defined by Ω. If u is any 1-form on $M(\Phi, \eta, \xi, g)$ such that du is equated by the second member of (2.10), then clearly $\mu^{-1}(u)$ defines an i.q.c.c.t.

From now on we shall be concerned with Sasakian manifold carrying a structure conformal vector field C.

Next let

$$\mathcal{O} = \text{vect.}\{e_i, \Phi e_i = e_{i^*}, e_0 = \xi \mid i = 1, \cdots, m\,;\ i^* = i + m\}$$

be an adapted local field of orthonormal frames on M and let

$$\mathcal{O}^* = \text{covect.}\{w^i, w^{i^*}, w^0 = \eta\}$$

be its associated coframe field.

Then the soldering form dp and É. Cartan's structure equations are :

$$dp = w^A \otimes e_A\,;\ A \in \{i, i^*, 0\} \tag{2.11}$$

and

$$\nabla e = \theta \otimes e, \tag{2.12}$$

$$d\omega = -\Theta \otimes \omega, \tag{2.13}$$

$$d\theta = -\theta \wedge \theta + \Theta, \tag{2.14}$$

respectively.

In the above equation θ (respectively Θ) is the local connection form in the bundle $\mathcal{O}(M)$. (respectively the curvature forms on M), and in terms of ω, the f.p.f. Ω is expressed by

$$\Omega = \sum_i \omega^i \wedge \omega^{i^*}, \tag{2.15}$$

Further since M is Sasakian, by (2.1),(2.3) and (2.12), one has

$$\theta^i_j = \theta^{i^*}_{j^*}, \qquad \theta^{i^*}_j = \theta^{j^*}_i \tag{2.16}$$

and by (2.16), we may check the formula $\nabla^2 \Phi X = -\flat(\Phi X) \wedge dp(X$ is an E.C. vector field) and

$$R(Z, Z') = -\Phi R(Z, Z')\Phi + Z \wedge Z' - \Phi Z \wedge \Phi Z'$$

(R is the curvature tensor).

Now in order to make simplifications, we set

$$\| C \|^2 = 2l, \flat(C) = \alpha, \flat(\Phi C) = \beta = i_C \Omega \tag{2.17}$$

and notice that one has

$$\beta = - < C, \nabla \xi > \tag{2.18}$$

Next, with the help of (2.1) and (2.12), we obtain from (2.7) that

$$dl = \rho\alpha - \lambda\beta, \tag{2.19}$$

$$d\eta(C) = \rho\eta - \beta \tag{2.20}$$

and

$$d\alpha = 2\lambda\Omega \Rightarrow \lambda = \text{const.}. \tag{2.21}$$

By (2.20), one gets at once

$$d\beta = d\rho \wedge \eta + 2\rho\Omega, \tag{2.22}$$

and by (2.17) the equation (2.22) implies

$$\mathcal{L}_C \Omega = 2\rho\Omega + d\rho \wedge \eta. \tag{2.23}$$

Since C is conformal vector field, then following the definition (0.2), we find the equation (2.23) that C is a quasi-biconformal vector field.

On the other hand, taking account of (2.4), one derives from (2.7) by covariant differentiation

$$\nabla^2 C = -(\lambda\eta - d\rho) \wedge p \tag{2.24}$$

The equation (2.24) proves that any structure conformal vector field on a Sasakian manifold is E.C.

Using (2.5), we find

$$\alpha = \flat(C) = \lambda\eta - d\rho, \tag{2.25}$$

and we notice that the equation (2.25) is consistent with (2.21).

Denote now by $\sum$ the exterior differential system which defines the structure conformal vector field C. Then, by (2.19),(2.20),(2.21),(2.22) and (2.25), we see that the *characteristic number* of $\sum$ (see [4]) are $r = 5, s_o = 3, s_1 = 2$. Consequently, following É.Cartan's test [4], we conclude that $\sum$ is in *involution* and depends on two arbitrary functions of one argument. Further, by (2.3) and (2.7), one derives

$$\nabla\Phi C = (\eta(C) - \lambda)dp + \rho\nabla\xi + d\rho \otimes \xi \tag{2.26}$$

Next, taking account of (2.25) and div Z =tr ∇Z, one finds

$$\mathrm{div}\Phi C = 2m(\eta(C) - \lambda) \tag{2.27}$$

We will outline the following property connected with this subject. First, by (2.25), the equation (2.22) becomes

$$d\beta = \eta \wedge \alpha + 2\rho\Omega, \tag{2.28}$$

and by (2.1) , one has

$$i_{\Phi C}\Omega = \flat(\Phi^2 C) = (\frac{2m+1}{m})\eta(C)\eta - \alpha \tag{2.29}$$

Then, taking account of (2.27), one may write

$$\mathcal{L}_{\Phi C}\beta = \frac{1}{m}\beta(\mathrm{div}\Phi C) = \frac{2m+1}{m}\frac{(\mathrm{div}\Phi C)\beta}{\mathrm{dim}M} \tag{2.30}$$

Hence, by definition (1.11), the equation (2.30) proves the following salient property : The structure conformal vector field C on a $(2m+1)$-dimensional Sasakian manifold M, turns out, under the action of Φ, to a self-conformal vector field of conformal weight $\frac{2m+1}{m}$

Next by (2.5), ΦC is also an E.C. vector field, i.e.

$$\nabla^2 \Phi C = -\flat(\Phi C) \wedge d\rho = -\beta \wedge dp. \tag{2.31}$$

Therefore, according to the general formula

$$S_{Z' \wedge Z} = \frac{< R(Z', Z)Z, Z' >}{\|Z'\|^2 \cdot \|Z\|^2 - < Z', Z >^2}; \qquad Z, Z' \in \mathcal{X}M,$$

and by (2.24) and (2.31), one finds that the ***sectional curvature*** $S_{\Phi C \wedge C}$ of ΦC and C is given by

$$S_{\Phi C \wedge C} = \frac{< R(\Phi C, C), \Phi C >}{\|C\|^2 \cdot \|\Phi C\|^2} = +1$$

Denote now by $\mathcal{D}_C = \{C, \Phi C, \xi\}$ the $\mathcal{D}$-distribution defined by $C, \Phi C$ and ξ. Then, if $X_C, X'_C \in \mathcal{D}_C$ are any vector fields of $\mathcal{D}_C$, it is easy to see by (2.1), (2.7) and (2.26), that one has $\nabla_{X'_C} X_C \in \mathcal{D}_C$ which expresses the fact that $\mathcal{D}_C$ is an *autoparallel foliation* (cf. [6]). On the other hand, since ξ, C and $\Phi C, \xi$ and E.C. vector fields, it follows, by linearity that any vector field X_C of $\mathcal{D}_C$ is E.C.

As a consequence of this fact and the results of [9], we conculde that the leaf M_C of $\mathcal{D}_C$ is an autoparallel submanifold of scalar curvature $+1$ of the Sasakian manifold $M(\Phi, \eta, \xi, g)$ under consideration.

Next, from (2.25) it follows

$$\operatorname{grad} \rho = \lambda \xi - C \tag{2.32}$$

and taking account of (2.7) one gets at once

$$\nabla \operatorname{grad} \rho = -\rho dp \tag{2.33}$$

which shows that grad ρ is a ***concurrent vector field*** [14]. From (2.33) one gets instantly

$$< \nabla_Z \operatorname{grad} \rho, Z' >= -\rho < Z, Z' > \tag{2.34}$$

Applying Obata's theorem (see (1.13)), we obtain that the Sasakian manifold under consideration enjoys the remarkable property to be isometric to a unit sphere in a (2m+2)-dimensional Euclidean space.

Thus, we proved the following theorem :

Theorem 2.1 Any Sasakian manifold $M(\Phi, \eta, \xi, g)$ which carries a structure conformal vector field C is foliated by autoparallel 3-dimensional submanifolds of scalar curvature +1 tangent to $C, \Phi C$ and ξ and is isometric to a unit sphere in a $(2m+2)$-dimensional Euclidean space. Further, one has the following properties :

(i) The existence of C is determined by an exterior differential system in involution.

(ii) Any C is an E.C. vector field and defines an infinitesimal quasi-conformal contact transformation of Ω

(iii) The vector field ΦC is self-conformal of conformal weight $\frac{2m+1}{m}$

3. Associated conformal vector fields of C

From (2.33) we get

$$\text{div grad}\rho = -(2m+1)\rho \tag{3.1}$$

Applying the general formula

$$\Delta\mu = -\text{div grad}\mu; \qquad \mu \in C^\infty M,$$

one finds

$$\Delta\rho = (2m+1)\rho \tag{3.2}$$

This proves that the conformal scalar ρ is an *eigenfunction* of Δ and has $2m+1$ as its associated *eigenvalue*. As a consequence of this fact, by (1.9), we may write

$$\mathcal{L}_C r = 2(2m(2m+1) - r)\rho, \tag{3.3}$$

where r denotes the scalar curvature of $M(\Phi, \eta, \xi, g)$.

In the view of the further discussions, we agree to set

$$\text{grad}\rho = C_1, \tag{3.4}$$

and since C_1 is concurrent, it also enjoys the properties to be conformal and E.C.

Therefore, by (1.10), the bracket $[C, C_1]$ is also a conformal vecto field: By (2.7) and (2.33), one has

$$[C, C_1] = \lambda(\Phi C - \rho\xi). \tag{3.5}$$

Noticing that

$$\Phi C - \rho\xi = [C, \xi] \tag{3.6}$$

and λ=const., we shall denote by C_2 the conformal vector field $[C, \xi]$.

Then with the help of (2.26) one derives

$$\nabla C_2 = \rho_2 dp, \tag{3.7}$$

where we have set (see (2.27))

$$\rho_2 = \eta(C) - \lambda = \frac{\text{div}\Phi C}{2m} \tag{3.8}$$

for the conformal scalar associated with C_2. By (2.20) and similar devices, one gets

$$\text{grad}\rho_2 = [\xi, C] = \rho\xi - \Phi C, \tag{3.9}$$

$$d\rho_2 = \rho\eta - \beta \tag{3.10}$$

and

$$\nabla\rho_2 = (2m+1)\rho_2, \tag{3.11}$$

which shows that ρ_2 enjoys the same properties as ρ and $\rho_1 = -\rho$, i.e. it is an eigenfunction of ∇ with $2m+1$ as the associated eigenvalue. One also easily finds $\nabla \mathrm{grad}\rho_2 = -\rho_2 dp$ which matches Obata's theorem (see (2.34)).

Next consider the conformal vector field

$$[C, C_2] = C_3 = -\rho C_2 - \eta(C)C_1 \tag{3.12}$$

By (2.20), (2.25), (2.33) and (3.7), the covariant differentiation of C_3 is experssed by

$$\nabla C_3 = \lambda \rho dp + \Phi C \lambda + \rho(C \wedge \xi) + \lambda(\xi \wedge \Phi C). \tag{3.13}$$

Since λ =const., the equation (3.13) shows that the conformal scalar $\rho_3 = \lambda\rho$ associated with C_3 satisfies also equation (3.2).

Finally consider the vector field $[C_1, C_2]$ and denote by $\rho_{[C_1,C_2]}$ its associated conformal scalar. Using computations, similar to those performed above, one fined $\rho_{[C_1,C_2]} = 0$, and this proves that the conformal vector field $[C_1, C_2]$ is a *Killing vector field*, i.e. $\mathcal{L}_{[C_1,C_2]}g = 0$. Effectively, setting $K = [C_1, C_2]$, using (3.7) and applying a straightforward calculation, one finds

$$\nabla K = -\Phi C \wedge C - \rho(C \wedge \xi). \tag{3.14}$$

It is easy to obtain from the equation (3.14) that the Killing equation is satisfied: $\mathcal{L}_K g = 0$. Besides, comparison of (3.13) and (3.14) gives

$$\nabla C_3 + \nabla K = \lambda \rho dp.$$

Therefore one may that the Killing vector field K is homologous exterior concurrent to the conformal vector field C_3.

In the following we agree to call $C_\nu, \nu = 1, 2, 3$(respectively K),the *associated conformal vector fields* (respectively the *associated Killing vector field*) of the structure vector field C.

As can be seen from (2.33), (3.2), (3.11) and (3.13),the conformal scalars ρ, ρ_1, ρ_2 and ρ_3 associated with these vector fields are eigenfunctions of Δ with the same eigenvalue equal to $\dim M$. Therefore one may say that these eigenfunctions define a 4-dimensional *eigenspace* $\varepsilon^4(M)$ of the eigenvalue dim M [12].

Further we will need some properties of conformal vector fields C, C_1 and C_3. We will discuss them now.

First of all, since we have seen that any vector field X_C of the distribution D_C is E.C.,it follows,by (1.3),(2.17),(3.4) and (3.6) that

$$(3.15)\qquad \begin{cases} \mathcal{R}(C,C) = 2m\|C\|^2 = 4ml, \\ \mathcal{R}(C_1,C_1) = 2m\|C_1\|^2 = 2m((\rho_2)^2 + 2l - \eta(C))^2), \\ \mathcal{R}(C_2,C_2) = 2m\|C_2\|^2 = 2m((\rho_1)^2 + 2l - \eta(C))^2), \end{cases}$$

where ρ_2 is expressed by (3.8) and $\rho_1 = -\rho$.

Next,by (2.7),(2.33) and (3.7), a short calculation gives

$$(3.16)\qquad \begin{cases} \|C\|^2 = (m+1)\rho^2 + 2m\lambda^2, \\ \|C_1\|^2 = (m+1)\rho_1^2, \\ \|C_2\|^2 = (m+1)\rho_2^2. \end{cases}$$

Assume now that $M(\Phi,\eta,\xi,g)$ is compact. Then, by reference to (1.12) and identifying $\mathcal{T}$ with C, C_1 and C_2 respectively, from (3.15) and (3.16) one finds the following integral formulas:

$$(3.17)\qquad \begin{cases} \int(\|C\|^2 + \lambda^2 - (1+2m)\rho^2) = 0, \\ \int(\|C\|^2 + \rho_2^2 - ((\eta(C))^2 - (1+2m)\rho^2) = 0, \\ \int(\|C\|^2 + \rho^2 - ((\eta(C))^2 - (1+2m)\rho^2) = 0. \end{cases}$$

The following theorem combines all results obtained in this section:

THEOREM 3.1. *Let C be a structure conformal vector field on a $(2m+1)$-dimensional Sasakian manifold $M(\Phi,\eta,\xi,g)$. By means of the Lie bracket, the vector field C generates three other conformal vector fields $C_\nu, \nu = 1,2,3$ called the associated conformal vector fields of C. The conformal scalars ρ, ρ_ν corresponding to these vector fields, define a 4-dimensional eigenspace $\varepsilon^4(M)$ of Δ of the eigenvalue dim $M = 2m+1$. If τ denotes the scalar curvature of M, then*

$$\mathcal{L}_C r = 2(2m(2m+1) - r)\rho,$$

and if M is compact, the integral formulas (3.17) hold.

4. Infinitesimal transformations defined by C and its associated vector fields

In this section we shall consider some infinitesimal transformations defined by the structure conformal vector field C. If ρ is the conformal scalar associated with C, we recall that one has

(4.1) $$(\mathrm{Hess}_\nabla \rho)(Z, Z') = g(Z, H_\rho Z'); H_\rho Z' = \nabla_{Z'} \mathrm{grad}\rho$$

and

(4.2) $$\nabla \mathrm{grad}\rho = -\rho dp.$$

First of all, by (1.7), (4.1) and (4.2), one fields after a short calculation that

(4.3) $$(\mathcal{L}_C R)(Z, Z', Z'') = 2\rho(Z \wedge Z')Z''; Z, Z', Z'' \in \mathcal{X}M.$$

Next, since C is E.C. which implies (cf.(1.3))

(4.4) $$\mathcal{R}(C, Z) = 2mg(C, Z),$$

by (1.8) and (3.2), one derives

(4.5) $$\mathcal{L}_C g(C, Z) = 2\rho g(C, Z).$$

Therefore we may say that C defines as infinitesimal conformal transformation (i.c.t.) of all of the tensors $g(C, Z)$. By a routine matter we may prove that similar results hold for the associated conformal vector fields C_ν of C.

Further, by the general formula (1.6) and (3.6), one readily finds

$$\mathcal{L}_C\eta = \rho\eta + \beta. \tag{4.6}$$

Now with the help of (4.6),(4.20) and the Lie derivative properties one writes

$$\mathcal{L}_{\eta(C)C}\eta = 2\eta(C)\rho\eta. \tag{4.7}$$

Hence the vector field $Y'_C = \eta(C)C$ defines an (i.c.t) of the structure 1-form η. Further, since $\mathrm{div}C = (2m+1)\rho$, one finds by (2.20)

$$\mathrm{div}Y'_C = 2(m+1)\eta(C)\rho, \tag{4.8}$$

and so (4.7) moves to

$$\mathcal{L}_{Y'_C}\eta = \frac{2m+1}{m+1}\left(\frac{\mathrm{div}Y'_C}{2m+1}\right)\eta, \tag{4.9}$$

which shows that Y'_C is of conformal weight $\frac{2m+1}{m+1}$ with respect to η[7].

We shall now investigate whether there exists another vector field $Y''_C \neq Y'_C$ of $\mathcal{D}_C$, which defines an (i.c.t) of η. With the help of (2.19) and (2.25) one finds by a straighgtforward calculation that

$$Y''_C = 2\lambda C + \rho\Phi C + 2l\xi, \tag{4.10}$$

and operating by ∇, one gets
(4.11)
$$\nabla Y''_C = \rho(\lambda+\eta(C))dp+\lambda)\Phi C\wedge\xi)+(\lambda\rho\eta+\rho\alpha-\lambda\beta)\otimes\xi+(2\lambda^2+\rho^2+2l)\nabla\xi.$$

From (4.11) one derives by taking tr $\nabla Y''_C$

$$\mathrm{div}Y''_C = 2(m+1)(\lambda+\eta(C))\rho, \tag{4.12}$$

and a standard calculation gives

$$\mathcal{L}_{Y''_C}\eta = \frac{2m+1}{m+1}\left(\frac{\mathrm{div}Y''_C}{2m+1}\right)\eta.$$

Hence as Y'_C, the vector field Y''_C is of conformal weight $\frac{2m+1}{m+1}$ with respect to η.

We shall now focus our attention on the distinguished conformal vector fields of $\mathcal{D}_C$. We recall that we have shown (see (2.33)) that C defines an infinitesimal quasi-conformal contact transformation of Ω. Form (2.32), (3.6) and (3.12) one derives

$$
\begin{cases}
\mathcal{L}_{C^1}\Omega = -\eta \wedge \alpha - 2\rho\Omega, \\
\mathcal{L}_{C^2}\Omega = -\eta \wedge \beta + 2(\eta(C) - \lambda)\Omega, \\
\mathcal{L}_{C^3}\Omega = -\lambda\eta \wedge \alpha + 2\lambda\rho\Omega.
\end{cases} \tag{4.13}
$$

which proves that all these vector fields are quasi-biconformal vector fields with respect to the pairing (η, Ω). Moreover, by a short calculation one derives from (2.23) and (4.13) that

$$d(\mathcal{L}_C\Omega) = 0, \qquad d(\mathcal{L}_{C^\nu}\Omega) = 0, \qquad \nu = 1, 2, 3.$$

Hence, following a well-known definition (see [1]), we may say that the structure 2-form Ω is a *relatively integral invariant* of the distinguished conformal vector fields of $\mathcal{D}_C$.

Finally since the Killing vector field $K = [C_1, C_2]$ can be expressed as

$$K = \rho_2 C_1 + \rho C_2; \qquad \rho_2 = \eta(C) - \lambda, \tag{4.14}$$

one quickly finds

$$i_K\Omega = \eta(C)d\eta(C) - \lambda dl \Rightarrow \mathcal{L}_K\Omega = 0.$$

According one may say that K defines an infinitesimal automorphism of the 2-form Ω. We arrived at the final theorem of our considerations:

THOEREM 4.1. *Let $M(\Phi, \eta, \xi, g)$ be a $(2m+1)$-dimensional Sasakian manifold carrying a structure conformal vector field C and let $\mathcal{D}_C$ be the autoparallel distribution defined by C. Let $C_\nu \in \mathcal{D}_C$, $\nu = 1, 2, 3$ be the associated conformal vector fields of C and let $K \in \mathcal{D}_C$ be the associated Killing vector field of C by the Lie bracket. Then*

(i) There are two vector fields $Y'_C, Y''_C \in \mathcal{D}_C$ which define infinitesimal conformal transformation of η, and both are of conformal weight $\frac{2m+1}{m+1}$ with respect to η.

(ii) *C and C_ν define infinitesimal conformal contact transformatons of the fundamental 2-form $\frac{1}{2}d\eta$, and $\frac{1}{2}d\eta$ is a relatively integral invariant of C and C_ν.*

(iii) *The Killing vector field K defines an infinitesimal automorphism of $\frac{1}{2}d\eta$.*

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