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ESSENTIALLY NORMAL ELEMENTS OF VON NEUMANN ALGEBRAS

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ABSTRACT. We prove that two essentially normal elements of a type II_∞ factor von Neumann algebra are unitarily equivalent up to the compact ideal if and only if they have the identical essential spectrum and the same index data. Also we calculate the spectrum and essential spectrum of a non-unitary isometry of von Neumann algebra.

In 1909 Hermann Weyl showed that a compact pertubation of self-adjoint operator leaves the spectrum invariant except for the isolated eigenvalues of finite multiplicity. In 1936 von Neumann supplied a striking converse: Two self-adjoint operators are unitarily equivalent modulo the compacts if and only if they have the same spectrum up to isolated eigenvalues of finite multiplicity. Then Berg[1] and Sikonia[11] extended this result to normal operator. Let H be a separable infinite dimensional Hilbert space, $L(H)$ the algebra of all bounded linear operators on H, $K(H)$ the two-sided ideal of compact operators in $L(H)$. and let π be the canonical homomorphism of $L(H)$ onto the Calkin algebra $L(H)/K(H)$. Then the above result about compact pertubations can be restated as follows: For two normal operators A and B in $L(H)$, $\pi(A)$ and $\pi(B)$ are unitarily equivalent in the Calkin algebra $L(H)/K(H)$ if and only if $\pi(A)$ and $\pi(B)$ have the same spectrum. The spectrum of $\pi(A)$ in the Calkin algebra is called the *esential spectrum* of the operator A.

An operator T is called *essentially normal* if $TT^* - T^*T$ is compact, or equivalently if $\pi(T)$ is normal in the Calkin algebra $L(H)/K(H)$. It is obvious that any normal operator is essentially normal but not every

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essentially normal operator is normal. Note however that any self-adjoint element in the Calkin algebra can be lifted to a self-adjoint operator in $L(H)$. An operator T is called *Fredholm* if (i) T has a closed range , (ii) the dimension of the kernel of T is finite, and (iii) the dimension of the kernel of T^* is finite. Then Atkinson theorem says that T is Fredholm if and only if $\pi(T)$ is invertible in the Calkin algebra. Then for each Fredholm operator T we define the index of T, $\operatorname{ind} T$, as

$$\operatorname{ind} T = \dim(\ker T) - \dim(\ker T^*),$$

where dim and ker denote the vector space dimension and the kernel of an operator, respectively. Then the celebrated Brown-Dougls-Fillmore theorem [4],[5] says that for two essentially normal operatprs N_1 and N_2 they are unitarily equivalent up to the compacts if and only if they have the same essential spectrum and for any λ not in the essential spectrum

$$\operatorname{ind}(N_1 - \lambda) = \operatorname{ind}(N_2 - \lambda).$$

Among many other things the above theorem is one of the earlier success of the beautiful theory initiated by Brown, Douglas, and Fillmore.

Let X be a compact metric space and $C(X)$ be the algebra of all complex- valued continuous functions on X. Then Ext X is defined as the set of C^*-algebra extensions

$$0 \to K(H) \to E \to C(X) \to 0$$

modulo a suitable equivalence relation, or equivalently as the unitary equivalence classes of unital *-monomorphisms of $C(X)$ into the Calkin algebra $L(H)/K(H)$. Ext X is an abelian group.

The extension theory has been generalized to a semifinite von Neumann factor. We need genralized notions of compactness and Freholmness of elements of von Neumann algebras (see more details [2],[3],[9],[10]). Let M be a semifinite II_∞ factor. Two projections p, q in M are said to be *equivalent* if there exists a partial isometry v in M such that $v^*v = p$ and $vv^* = q$. A projection p in M is called *finite* if p is not equivalent to any proper subprojection of p. If a projection p is not finite, then p is called *infinite*. Let $P(M)$ be the set of all projections in M. Then there is a function dim: $P(M) \to [0, \infty]$ which possesses the similar properties of the usual dimension function of vector spaces. For instance, a projection p is finite if and only if $\dim(p) < \infty$ and a projection p is equivalent to a subprojection of q if and only if $\dim(p) \leqq \dim(q)$. And a projection p is infinite if and only if $\dim(p) = \infty$.

Let $K(M)$ be the norm closed two-sided *-subalgebra of M generated by all finite projections in M. This ideal $K(M)$ resembles the usual compact ideal $K(H)$ of $L(H)$. An element $x \in K(M)$ is called *compact* and $K(M)$ is called the *compact ideal* of M. Let π be the canonical homomorphism of M onto the generalized Calkin algebra $M/K(M)$. An element $t \in M$ is called *essentially normal* if $t^*t - tt^*$ is compact, or equivalently if $\pi(t)$ is normal in the generalized Calkin algebra $M/K(M)$. The following definition is due to Breuer [2]. An element $t \in M$ is called *Fredholm (relative to M)* if

(i) $N_t = \sup\{e \in P(M) | te = 0\}$ is finite

(ii) there is a finite projection p such that $R_t = \inf\{e \in P(M) | et = t\}$ is a subprojection of $1 - p$.

The projection N_t and R_t are called the *null* and the *range* projection of t, respectively. Then the second condition (ii) implies that N_{t^*} is finite but it does not mean that t has a closed range when M is considered to be acting on a Hilbert space H. For each Fredholm element t in M we define the index of t, $\operatorname{ind} t$ (we use the same notation), as follows:

$$\operatorname{ind} t = \dim(N_t) - \dim(N_{t^*}).$$

Furthermore Breuer [2],[3] extended the classical Atkinson theorem to the II_∞ factor case. That is, an element t in M is Fredholm if and only if $\pi(t)$ is invertible in the generalized Calkin algebra $M/K(M)$. Also Zsido [12] extended the Weyl-von Neumann theorem to II_∞ factor cases.

We will prove a similar theorem of Brown-Douglas-Fillmore about the unitary equivalence of essentially normal operators.

We briefly dicuss the extension group of $C(X)$ relative to a II_∞ factor M developed by Fillmore [7] and Cho [6]. A unital *-monomorphism $\tau : C(X) \to M/K(M)$ is called an *extension*. Let $\mathrm{Ext}^M X$ be the unitary equivalence classes of extensions. Then $\mathrm{Ext}^M X$ is an abelian group. Let SX denote the suspension of X. Then there is a natural transformation

$$\gamma_\infty : \mathrm{Ext}^M X \to \mathrm{Hom}(\widetilde{K}(SX), \mathbb{R}).$$

Note that $\widetilde{K}(SX) = \text{inj lim}[X, GL_n]$,where $[X, GL_n]$ denotes the homotopy classes of X into the group GL_n of invertible $n \times n$ matries. The natural transformation γ_∞ is given as follows: For a given extension τ and a continuous function $f : X \to Gl_n$, the map γ_∞ is induced by

$$(\tau, f) \to \text{ind}(\tau_n(f)),$$

where

$$\begin{aligned}\tau_n(f) &= (\tau(f_{ij})) \in M/K(M) \otimes M_n(C) \\ &\cong M \otimes M_n(C)/K(M \otimes M_n(C)) \cong M/K(M).\end{aligned}$$

It can be seen that by using the index theory developed by Breuer [4] this construction depends only on the homotopy class of f and the equivalence class of τ, it respects the obvious inclusion of GL_n into GL_{n+1} by $A \to \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$, and it is a homomorphism. Furthermore, the natural transformation $\tau_\infty : \text{Ext}^M X \to \text{Hom}(\widetilde{K}(SX), \mathbb{R})$ is an isomorphism for any compact metric space X. For any compact subset X of the complex plane, note that $\widetilde{K}(SX) \cong \pi^1(X)$, where $\pi^1(X)$ denotes the first cohomology group of X. In this spacial case, the natural transformation γ_∞ is given by $\gamma_\infty(\tau)(f) = \text{ind}(\tau(f))$, for each $\tau \in \text{Ext}^M X$ and each $f \in \pi^1(X)$.

We are now ready to state and prove the complete invariance of essentially normal elements of a type II_∞ factor M. The theorem and its proof are exactly same as the classical ones.

THEOREM 1. *Let M be a type II_∞ factor. Then two essentially normal elements n_1 and n_2 are unitarily equivalent up to the compact ideal $K(M)$ of M if and only if they have the identical essential spectrum X and $ind(n_1 - \lambda) = ind(n_2 - \lambda)$ for all λ not in X.*

PROOF. Suppose that two essentially normal elements n_1 and n_2 are unitarily equivalent up to the compacts. Then there exists a unitary u in M such that $n_2 - u^* n_1 u \in K(M)$. Thus $\pi(n_2) = \pi(u^*)\pi(n_1)\pi(u)$. Hence $\pi(n_1)$ and $\pi(n_2)$ have the identical spectrum. Therefore n_1 and n_2 have the identical essential spectrum. Furthermore, since the index is invariant under the compact pertubations, we have $\text{ind}(n_2 - \lambda) =$

$\text{ind}(u^*(n_1 - \lambda)u)$ for all $\lambda \notin X$. Since $\text{ind}(u) = 0$ for any unitary u in M and $\text{ind}(st) = \text{ind}(s) + \text{ind}(t)$ for any Fredholm elements s and t, we have $\text{ind}(n_2 - \lambda) = \text{ind}(n_1 - \lambda)$ for all λ not in the essential spectrum X.

Conversely, let τ_i be the extensions induced by the spectral theorem of normal elements $\pi(n_i)$ in $M/K(M)$ for $i = 1, 2$. Thus $\tau_i(z) = \pi(n_i)$ for $i = 1, 2$, where z denotes the identity function on X. We will show $\text{ind}(\tau_2(f)) = \text{ind}(\tau_1(f))$ for any invertible function $f \in C(X)$. If we do this, then $\gamma_\infty(\tau_2) = \gamma_\infty(\tau_1)$. Then since γ_∞ is an isomorphism, τ_1 and τ_2 are the same extensions. This means that there exists a unitary u in M such that $\tau_2(z) = \pi(u^*)\tau_1(z)\pi(u)$. Since $\tau_i(z) = \pi(n_i)$ for $i = 1, 2$, we have $\pi(n_2) = \pi(u^*)\pi(n_1)\pi(u)$, i.e., $n_2 - u^*n_1u \in K(M)$. Thus it remains to show that $\text{ind}(\tau_2(f)) = \text{ind}(\tau_1(f))$ for any invertible function $f \in C(X)$. Note that $\pi^1(X)$ is the free abelian group with one generator, say, $z - \lambda$ for each bounded connected component O of the complement of X in the complex plane and a $\lambda \in O$. For such a generator, we have

$$\text{ind}(\tau_2(z - \lambda)) = \text{ind}(n_2 - \lambda) = \text{ind}(n_1 - \lambda) = \text{ind}(\tau_1(z - \lambda)).$$

Hence $\gamma_\infty(\tau_1)$ and $\gamma_\infty(\tau_2)$ agree on the generator of $\pi^1(X)$. Thus $\gamma_\infty(\tau_2) = \gamma_\infty(\tau_1)$ on $\pi^1(X)$. This completes the proof.

Recall that an isometry U on a Hilbert space H is called a *unilateral shift* if there exists an orthonormal basis $\{e_n\}_{n=1}^{\infty}$ such that $U(e_n) = e_{n+1}$ for all n. Then the spectrum of the unilateral shift U is the closed unit disc $\overline{\mathbb{D}}$ and the essential spectrum of U is the unit circle $\mathbb{T}$. Moreover, for any complex number λ with $|\lambda| < 1$, there exists a one-dimensional projection P such that $U^*P = \overline{\lambda}P$. In passing, we mention that Fillmore,Stampli, and Williams [8] proved that for a hyponormal element a in the Calkin algebra $L(H)/K(H)$, λ is in the spectrum of a if and only if there is a non-zero projection p such that $a^*p = \lambda p$. We generalize the theorem about the spectrum of the unilateral shift to general isometries in a type II_∞ factor M.

THEOREM 2. *Let u be an isometry in a type II_∞ factor M such that $1 - uu^*$ is a non-zero finite projection. Then the essential spectrum of u is the unit circle $\mathbb{T}$ and for any complex number λ with $|\lambda| < 1$ there exists a non-zero projection p in M such that $u^*p = \overline{\lambda}p$ and hence the spectrum of u is the closed unit disc $\overline{\mathbb{D}}$.*

PROOF. We first show that the essential spectrum $\sigma_e(u)$ of u is equal to the unit circle $\mathbb{T}$. Since $\pi(u)$ is a unitary in the generalized Calkin algebra $M/K(M)$, we have $\sigma_e(u) \subset \mathbb{T}$. If $\sigma_e(u) \neq \mathbb{T}$, then the complement of $\sigma_e(u)$ is connected and hence $\operatorname{ind}(u-\lambda) = 0$ for any $\lambda \notin \sigma_e(u)$. In particular, $\operatorname{ind} u = 0$. But $\operatorname{ind} u = \dim(1 - u^*u) - \dim(1 - uu^*) = -\dim(1 - uu^*) \neq 0$, which is a contradiction. Hence $\sigma_e(u) = \mathbb{T}$.

Secondly, for any complex number λ with $|\lambda| < 1$, since the index is invariant in the connected components of the set of all Fredholm elements of M, we have $\operatorname{ind}(u - \lambda) = \operatorname{ind} u$. But $\operatorname{ind} u \neq 0$. Therefore $\dim N_{(u-\lambda)^*} \neq 0$. Hence there exists a non-zero projection p in M such that $(u-\lambda)^* p = 0$, which in turn implies that $u^* p = \overline{\lambda} p$. Thus $\overline{\lambda} \in \sigma(u^*)$. Therefore, $\lambda \in \sigma(u)$. Since $\sigma(u) \subset \overline{\mathbb{D}}$ and $\sigma(u)$ is a compact subset, we have $\sigma(u) = \overline{\mathbb{D}}$. This completes the proof.

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