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A GOLUSIN SEMI-VARIATION

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ABSTRACT. We use a semi-variational method to obtain necessary condition for a linear functional L to attain its extrema at certain elementary products. This is applied to obtain an answer to the long-standing question of determining explicit extreme points of the normalised Spirallike functions of order α.

1. Introduction

Let E_g be the class of regular functions on the open unit disc $\triangle$ given by Stieltjes integrals of the form $f(z) = \int_{-\pi}^{\pi} g(z,t)d\psi(t)$ where $\psi(t)$ is an increasing function on $[-\pi, \pi]$ with $\psi(-\pi) = 0$ and $\psi(\pi) = 1$ and $g(z,t)$ is continuous on $\triangle \times [-\pi, \pi]$ and regular on $\triangle$ for each $t \in [-\pi, \pi]$.

The variatioal principle of Golusin [2] asserts that for $-\pi \leq t_1 < t_2 \leq \pi$

(i) the function $f_*(z) = f(z) + \lambda \int_{t_1}^{t_2} \frac{\partial g}{\partial t}(z,t)|\psi(t) - C|dt$ lies in E_g for all real values of λ in some open interval about 0, where C is a constant independent of z, t and λ, and

(ii) if $\psi(t)$ has jumps at the points t_1 and t_2 then the function

$$f_{**}(z) = f(z) + \lambda(g(z,t_2) - g(z,t_1))$$

also lies in E_g for all real values of λ in some open interval about 0.

Our results rely on the additional fact that

(iii) if $\psi(t)$ has a jump at t_1 and t is any other point of $[-\pi, \pi]$ then

$$f_+(z) = f(z) + \epsilon(g(z,t) - g(z,t_1))$$

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lies in E_g for all real ϵ with $0 < \epsilon \leq \psi(t_1^+) - \psi(t_1^-)$.

When a continuous functional $L(f)$ attains an extremum at a function $f(z)$ in E_g represented by $\psi(t)$, the semi-variation (iii) yields necessary conditions on $f(z)$ which cannot be obtained by the use of (i) and (ii) alone. Clearly (ii) follows from (iii).

In the present paper we consider applications to the set $Sp(\alpha)$ of spirallike functions of order α, where $0 < |\alpha| < \frac{\pi}{2}$. These include the set Π_n of all products $z\Pi_{k=1}^{n}(1-u_k z)^{-ma_k}$ where the n points u_k are distinct with $|u_k| = 1$, $m = 1 + e^{-2i\alpha}$, $a_k > 0$ for $k = 1, \cdots, n$ and $\sum_{k=1}^{n} a_k = 1$.

Let $\Pi = \cup_{n=1}^{\infty}\Pi_n$. We regard $Sp(\alpha)$ as a subset of the space $\mathcal{A}_0$ of analytic functions on $\triangle$ vanishing at 0, with the topology of uniform convergence on compact subsets of $\triangle$.

It is well-known that the functions Π_1 are extreme points of the convex hull $co\ Sp(\alpha)$, as proved in Example 1 below. In the case $\cos^{-1}(\frac{1}{3}) < |\alpha| < \frac{\pi}{2}$, we provide an answer to the long-standing question [5] of determining explicit extreme points of $Sp(\alpha)$ which lie in $\Pi \backslash \Pi_1$. The existence of such functions for all values $0 < |\alpha| < \frac{\pi}{2}$ was shown in [4], using a different method based on a result in [6]. Here we prove that the Π_2 function $z(1-z^2)^{-\frac{m}{2}}$ is an extreme point of $Sp(\alpha)$ for sufficiently large $|\alpha|$.

We use the method of [7] to consider the support sets of continuous linear functionals of the form

$$L(f) = \text{Re} \sum_{j=0}^{N} \left(\frac{d^j}{dz^j} b_j(z) f(z) \right)_{z=\zeta}$$

with $|\zeta| < 1$, where $b_j(z)f(z)$ is regular on a neighbourhood of ζ for $j = 0, \cdots, N$ for all $f \in \mathcal{A}_0$. We extend Theorem 14 of [7] to include case where $\zeta = 0$ or $b_j(z)$ is not constant, such as the linear functional $L(f) = \left(\frac{d^N}{dz^N} \frac{f(z)}{z} \right)_{z=0}$.

This shows that these linear functionals attain their extrema on $Sp(\alpha)$ only at points of Π. We then use the semi-variation (iii) to get necessary conditions for L to attain its maximum at specific products in $p \in \Pi_n$.

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2. Applications of the semi-variational principle to linear optimisation

The method has wide applicability to many of the standard classes of univalent functions, using the following proposition. We consider here the maximisation of finite length linear functionals on the class of spirallike function.

PROPOSITION 1. *Let E_g be the family of functions G given by the integral $f(z)=\int_{-\pi}^{\pi} g(z,t)d\psi(t)$ where $\psi(t)$ is an increasing function on $[-\pi,\pi]$ with $\psi(-\pi)=0$, $\psi(\pi)=1$ and g is continuous on $\triangle\times[-\pi,\pi]$ and regular on $\triangle$ for each $t\in[-\pi,\pi]$.*

If $\psi(t)$ has a jump at t_1 and t is any point of $[-\pi,\pi]$ then for $0<\epsilon\le\psi(t_1^+)-\psi(t_1^-)$ the function $f_+(z)=f(z)+\epsilon(g(z,t)-g(z,t_1))$ lies in E_g.

PROOF. When $-\pi\le t<t_1\le\pi$ and ϵ is as above, the function $\psi_1(s)=\psi(s)+\epsilon\mathcal{X}_{[t,t_1]}(s)$ is increasing and normalised on $[-\pi,\pi]$ and $f_+(z)=\int_{-\pi}^{\pi} g(z,s)d\psi(s)=f(z)+\epsilon\int_{-\pi}^{\pi} g(z,s)d\mathcal{X}_{[t,t_1]}(s)=f(z)+\epsilon(g(z,t)-g(z,t_1))$. When $-\pi\le t_1<t\le\pi$, the function $\psi_2(s)=\psi(s)-\epsilon\mathcal{X}_{[t_1,t]}(s)$ yields the same result.

The variational principle (ii) follows by choosing $t=t_p$, $t_1=t_q$ and $t=t_q$, $t_1=t_p$ respectively, where $\psi(t)$ has jumps at the distinct points t_p, t_q and $|\lambda|<\max\left(\psi(t_p^+)-\psi(t_p^-),\psi(t_q^+)-\psi(t_q^-)\right)$.

The spirallike functions of order α, where $0<|\alpha|<\frac{\pi}{2}$, are the analytic functions $f(z)$ satisfying $\mathrm{Re}\frac{e^{i\alpha}zf'(z)}{f(z)}>0$ on the open unit disc $\triangle$, with $f(0)=0$ and $f'(0)=1$.

They are characterised by the representation

$$f(z)=z\ \exp\left(-m\int_{-\pi}^{\pi}\log(1-ze^{-it})d\psi(t)\right)$$

where $m=1+e^{-2i\alpha}$. The same representation characterises S_β, the starlike functions of order β when m is replaced by the real constant $2-\beta$ where $0<\beta<2$.

Let Π_n denote the subset of $Sp(\alpha)$ consisting of all products $z\Pi_{k=1}^{n}(1-u_kz)^{-ma_k}$ where the n points u_k are distinct with $|u_k|=1$, $a_k>0$ for $k=1,\cdots,n$ and $\sum_{k=1}^{n}a_k=1$. Let $\Pi=\cup_{n=1}^{\infty}\Pi_n$.

DEFINITION. A linear functional of the form

$$L(f) = \text{Re} \sum_{j=0}^{N} \left(\frac{d^j}{dz^j} b_j(z) f(z) \right)_{z=\zeta}$$

with $|\zeta| < 1$ and where $b_j(z)f(z)$ is regular on a neighbourhood of ζ for $j = 0, \cdots, N$ and all $f \in \mathcal{A}_0$ will be called a finite length functional of length N.

The following result extends Theorem 14 of [7] to include cases such as the linear functional $L(f) = \left(\frac{d^N}{dz^N} \frac{f(z)}{z} \right)_{z=0}$, where we remove the singularity of $\frac{f(z)}{z}$ at $z = 0$.

THEOREM 1. *Every finite length functional L of length N on $\mathcal{A}_0$ is maximised on $Sp(\alpha)$ only at points of $\cup_{n=1}^{N+1} \Pi_n$ if $\zeta \neq 0$ and only at points of $\cup_{n=1}^{N} \Pi_n$ if $\zeta = 0$.*

If L attains its maximum at $p(z) = z\Pi_{k=1}^{n}(1 - u_k z)^{-m a_k}$ then $u_k = e^{-it_k}$ where the t_k are among those zeros of the function

$$Q(t) = Re\, m \sum_{j=0}^{N} \left(\frac{d^j}{dz^j} \frac{i z b_j(z) p(z)}{e^{it} - z} \right)_{z=\zeta}$$

which satisfy $Q'(t) < 0$.

PROOF. Let the finite length functional L attain its maximum on $Sp(\alpha)$ at the function $f(z)$, where L is as above and f is represented by the increasing function ψ.

Using Golusin's variation (i), for each pair t_1, t_2 with $-\pi \leq t_1 \leq t_2 \leq \pi$, there exists a constant C independent of z and t such that for all real λ in an open interval containing 0 the function

$$\begin{aligned} f_*(z) &= f(z) \exp(-m\lambda \int_{t_1}^{t_2} \frac{iz}{e^{it} - z} |\psi(t) - C| dt) \\ &= f(z) - \lambda m \int_{t_1}^{t_2} \frac{i z f(z)}{e^{it} - z} |\psi(t) - C| dt + O(\lambda^2) \end{aligned}$$

lies in $Sp(\alpha)$. Applying L to f_* yields

$$L(f_*) = L(f) - L\Big(\lambda m \int_{t_1}^{t_2} \frac{izf(z)}{e^{it}-z}|\psi(t) - C|dt\Big) + O(\lambda^2)$$

$$= L(f) - \lambda \int_{t_1}^{t_2} \text{Re } m \sum_{j=0}^{N} \Big(\frac{d^j}{dz^j}\frac{izb_j(z)f(z)}{e^{it}-z}\Big)_{z=\zeta} |\psi(t) - C|dt + O(\lambda^2)$$

$$= L(f) - \lambda \int_{t_1}^{t_2} Q(t)|\psi(t) - C|dt + O(\lambda^2).$$

As in [7],

$$Q(t) = \text{Re } m \sum_{j=0}^{N} \Big(\frac{d^j}{dz^j}\frac{izb_j(z)f(z)}{e^{it}-z}\Big)_{z=\zeta} = \text{Re } \sum_{j=1}^{N+1} A_j(e^{it} - \zeta)^{-j}$$

$$= e^{-i(N+1)t}|e^{it} - \zeta|^{-2N-2} \sum_{j=0}^{2N+2} B_j e^{ijt}$$

which has at most $2N + 2$ zeros in $[-\pi, \pi)$.

When $\zeta = 0$ the leading term is

$$A_{N+1} = \lim_{z\to 0}(-1)^N N!\, izf(z)b_N(z)(e^{it} - z)^{-N-1} = 0$$

since $b_N(z)f(z)$ is regular at $z = 0$. So in this case $Q(t)$ has at most $2N$ zeros in $[-\pi, \pi)$.

Since $L(f) \geq L(f_*)$ for arbitrarily small $|\lambda|$, the coefficient of λ vanishes giving $\int_{t_1}^{t_2} Q(t)|\psi(t) - C|dt = 0$.

Now $Q(t)$ is continuous so that $\psi(t)$ is constant on any interval $[t_1, t_2]$ where Q does not change sign, and must be a step function with at most $2N + 2$ jump points at zeros of $Q(t)$, or at most $2N$ if $\zeta = 0$.

Let t_1 be any jump point of $\psi(t)$ and let t be any other point of $[-\pi, \pi)$. Using the semi-variation f_+ for small positive ϵ we get

$$L(f) \geq L(f_+) = L(f) + \epsilon L(\log(1 - ze^{-it}) - \log(1 - ze^{-it_1})).$$

Setting

$$R(t) = L(\log(1 - ze^{-it})) = \text{Re} \sum_{j=0}^{N} \Big(\frac{d^j}{dz^j} b_j(z) \log(1 - ze^{-it}) \Big)_{z=\zeta},$$

we have $R(t_1) \geq R(t)$ for all $t \in [-\pi, \pi]$. Thus $R(t)$ takes the same maximum value at all the jumps of $\psi(t)$. At these points $R'(t) = Q(t) = 0$ and $R''(t) = Q'(t) < 0$, so there can be at most $N + 1$ zeros of $Q(t)$ which are also local maxima of $R(t)$, or at most N in the case $\zeta = 0$. □

We now need only consider finite products of the type $p(z) = z\Pi_{k=1}^{M}(1 - u_k z)^{-ma_k}$ which correspond to discrete probability measures $\mu = \sum_{k=1}^{M} a_k \delta_{u_k}$.

THEOREM 2. *Let the finite length functional L of length $N \geq 1$ attain its maximum at the point $p(z) = z\Pi_{k=1}^{M}(1 - u_k z)^{-ma_k}$ of $Sp(\alpha)$.*

Then either (a) $p \in \cup_{n=2}^{N+1}\Pi_n$ *and* $\sum_{j=0}^{N}\Big(\frac{d^j}{dz^j} z b_j(z) p(z)\Big)_{z=\zeta} = 0$

or (b) $p(z) = z(1 - u_1 z)^{-m}$, *with* $|u_1| = 1$,

and $mu_1 \sum_{j=0}^{N}\Big(\frac{d^j}{dz^j} z b_j(z) p(z)\Big)_{z=\zeta} \geq 0$.

In case (a), if $\zeta = 0$ *then* $p \in \cup_{n=2}^{N}\Pi_n$.

PROOF. The upper limits on n are given by Theorem 1.

Let L attain its maximum on $Sp(\alpha)$ at the point p with representing measure $\mu = \sum_{k=1}^{M} a_k \delta_{u_k}$.

Let $0 < \epsilon < \min(a_1, \cdots, a_n)$ and let p^+ be the element of Π represented by the measure $\nu = \mu + \epsilon(\delta_u - \delta_{u_q})$ where u_q is in the support of μ and $|u| = 1$.

Since $p^+(z) = z(1 - uz)^{-m\epsilon}(1 - u_q z)^{m\epsilon}\Pi_{k=1}^{M}(1 - u_k z)^{-ma_k} = p(z)(1 + \epsilon mz(u - u_q) + O(\epsilon^2))$ we have

$$L(p^+) = L(p) + \epsilon(L(muzp(z)) - L(mu_q zp(z))) + O(\epsilon^2) \leq L(p)$$

for arbitrarily small values of $\epsilon > 0$, giving $L(m(u - u_k)zp(z)) \leq 0$ so that

$$\text{Re}\, m(u - u_k) \sum_{j=0}^{N} \Big(\frac{d^j}{dz^j} z b_j(z) p(z) \Big)_{z=\zeta} \leq 0$$

for all $|u| = 1$.

Now if $u_k = e^{i\alpha_k}$, $\arg(u - u_k)$ takes all values in $(\frac{\pi}{2} + \alpha_k, \frac{3\pi}{2} + \alpha_k)$, implying that

$$m \sum_{j=0}^{N} \left(\frac{d^j}{dz^j} z b_j(z) p(z) \right)_{z=\zeta} = A e^{-i\alpha_k}$$

where $A \geq 0$.

When $n = 1$ we have $p(z) = z(1 - u_1 z)^{-m}$ and $m u_1 \sum_{j=0}^{N} \left(\frac{d^j}{dz^j} z b_j(z) p(z) \right)_{z=\zeta} \geq 0$.

When $p \in \Pi_n$ for $n \geq 2$ then $e^{i\alpha_k}$ can take two distinct values, implying that

$$\sum_{j=0}^{N} \left(\frac{d^j}{dz^j} z b_j(z) p(z) \right)_{z=\zeta} = 0.$$

$\square$

COROLLARY 1. *(a) The linear functional* $L(f) = Re\, \overline{m} \left(\frac{d^N}{dz^N} f(z) \right)_{z=\zeta}$ *attains its maximum on* $Sp(\alpha)$ *at* $p \in \Pi_k$*, where* $2 \leq k \leq N+1$ *and* $\zeta p^{(N)}(\zeta) + N p^{(N-1)}(\zeta) = 0$ *or* $k = 1$, $p(z) = z(1 - uz)^{-m}$ *and* $u(\zeta p^{(N)}(\zeta) + N p^{(N-1)}(\zeta)) \geq 0$.

(b) The linear functional $L(f) = Re\, \overline{m} \left(\frac{d^N}{dz^N} \frac{f(z)}{z} \right)_{z=\zeta}$ *attains its maximum on* $Sp(\alpha)$ *at* $p \in \Pi_k$*, where* $2 \leq k \leq N+1$ *and* $p^{(N)}(\zeta) = 0$ *or* $k = 1, p(z) = z(1 - uz)^{-m} \in \Pi_1$ *and* $u p^{(N)}(\zeta) \geq 0$.

(c) When $\zeta = 0, k$ *satisfies* $1 \leq k \leq N$ *in (a) and (b).*

COROLLARY 2. *Any non-constant finite length linear functional on* $\mathcal{A}_0$ *which attains its maximum on* $Sp(\alpha)$ *at points of* Π_1 *does so only at a finite number of points.*

PROOF. The function

$$g(u) = m u_1 \sum_{j=0}^{N} \left(\frac{d^j}{dz^j} z b_j(z) \frac{z}{(1 - uz)^m} \right)_{z=\zeta}$$

is regular on $\triangle$ so by Theorem 16 of [1] the image of the unit circle intersects the real axis in a finite number of points unless $g(u)$ is constant.

3. Examples

We now give examples showing the use of the above results in specific cases.

EXAMPLE 1. For each v with $|v| = 1$ the linear functional $L(f) = \operatorname{Re} \overline{mv}\left(\frac{d}{dz}\frac{f(z)}{z}\right)_{z=0}$ attains its maximum on $Sp(\alpha)$ at functions $p(z) = z(1-uz)^{-m}$ with $|u| = 1$ and which satisfy $\overline{v}u \geq 0$.

That is, L is maximised uniquely at the function $p(z) = z(1-vz)^{-m}$, which proves that all points of Π_1 are extreme points of $Sp(\alpha)$.

EXAMPLE 2. The linear functional $L(f) = \operatorname{Re} \overline{m}\left(\frac{d^2}{dz^2}\frac{f(z)}{z}\right)_{z=0}$ attains its maximum on $Sp(\alpha)$ on $\Pi_1 \cup \Pi_2$.

If the maximum is attained on Π_1 at a function $p(z) = z(1-uz)^{-m}$ with $|u| = 1$, then u must satisfy

$$u\left(\frac{d^2}{dz^2}\frac{z}{(1-uz)^m}\right)_{z=0} = 2mu^2 \geq 0$$

This requires that $u^2 = \frac{\overline{m}}{|m|}$, so that $u = \pm e^{\frac{1}{2}i\alpha}$ and the maximum value is then

$$L(p) = \operatorname{Re} m\overline{m}u^2(m+1) = |m|(|m|^2 + \operatorname{Re} m) = 12\cos^3 \alpha,$$

where $m = 1 + e^{-2i\alpha}$ and $|m| = 2\cos\alpha$.

If the maximum is attained on Π_2 at a function

$$q(z) = z(1-u_1z)^{-mt_1}(1-u_2z)^{-mt_2}$$

with $|u_1| = 1$, $|u_2| = 1$, $t_1 > 0$, $t_2 > 0$ and $t_1 + t_2 = 1$ we require

$$\left(\frac{d^2}{dz^2}q(z)\right)_{z=0} = 2m(t_1u_1 + t_2u_2) = 0.$$

This can only happen when $u_1 = -u_2$ and $t_1 = t_2 = \frac{1}{2}$.

The maximum value is then given by

$$L(q) = \mathrm{Re}\, \overline{m}\Big(\frac{d^2}{dz^2}(1 - u_1^2 z^2)^{-\frac{m}{2}}\Big)_{z=0} = \mathrm{Re}\, m\overline{m}u_1^2 \le |m|^2 = 4\cos^2\ \alpha,$$

which is attained on Π_2 only at the function $z(1 - z^2)^{-\frac{m}{2}}$.

There are thus three cases.

(i) $\cos\alpha > \frac{1}{3}$. $L(f)$ attains its maximum on $Sp(\alpha)$ uniquely at the two functions $z(1 - uz)^{-m}$ in Π_1 with $u = \pm e^{\frac{1}{2}i\alpha}$.

(ii) $\cos\alpha = \frac{1}{3}$. $L(f)$ attains its maximum on $Sp(\alpha)$ at the two functions $z(1 - uz)^{-m}$ in (i), at the function $z(1 - z^2)^{-\frac{m}{2}}$ in Π_2 and only at these.

(iii) $\cos\alpha < \frac{1}{3}$. $L(f)$ attains its maximum on $Sp(\alpha)$ uniquely at the function $z(1 - z^2)^{-\frac{m}{2}}$.

Cases (ii) and (iii) provide an answer to the long-standing question of determining explicit extreme points of $Sp(\alpha)$ which lie in $\Pi\backslash\Pi_1$, in the case $\cos^{-1}(\frac{1}{3}) < |\alpha| < \frac{\pi}{2}$. The existence of such functions for all values $0 < |\alpha| < \frac{\pi}{2}$ was shown in [4].

THEOREM 3. *The function* $z(1 - z^2)^{-\frac{m}{2}}$ *is an extreme point of the spirallike functions* $Sp(\alpha)$ *when* $\cos^{-1}(\frac{1}{3}) \le |\alpha| < \frac{\pi}{2}$.

PROOF. Case (iii), above, gives the result in the case of strict inequality.

When $3\ \cos\alpha = 1$, we must show that $z(1-z^2)^{-\frac{m}{2}}$ does not lie on the line joining $z(1 \pm e^{\frac{1}{2}i\alpha}z)^{-m}$. This cannot occur because the first function would have to have singularities at $\pm e^{\frac{1}{2}i\alpha}$.

EXAMPLE 3. Let $|\zeta| < 1$ and $\zeta \ne 0$. The linear functional $L(f) = \mathrm{Re}\, \overline{m}\Big(\frac{d}{dz}\frac{f(z)}{z}\Big)_{z=\zeta}$ does not attain its maximum on $Sp(\alpha)$ at the function $p(z) = z(1 - z^2)^{-\frac{m}{2}}$.

This is because $p'(\zeta) = (1 + e^{-2i\alpha}\zeta^2)(1 - \zeta^2)^{-\frac{m}{2}-1} \ne 0$ when $|\zeta| < 1$.

EXAMPLE 4. If for $k \ge 2$ the linear functional $L(f) = \mathrm{Re}\, \overline{m}\Big(\frac{d^k}{dz^k}\frac{f(z)}{z}\Big)_{z=\zeta}$ attains its maximum on $Sp(\alpha)$ at a function $z(1 - uz)^{-m}$ in Π_1 then

$$g(u) = m(m + 1)\cdots(m + k - 2)u^{k-1}(k + (m - 1)u\zeta)(1 - u\zeta)^{-m-k} \ge 0.$$

By Corollary 2, this holds for only a finite number of values of u, since L is non-constant.

NOTE. Many similar calculations are possible. All the general results above are available for application to the family of starlike functions of order β, where m may be replaced by the real constant $2-\beta$ with $0<\beta<2$.

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