Comm. Korean Math. Soc. **10** (1995), No. 3, pp. 603–607

A SPECIAL DECOMPOSITION OF REGULAR $*$–SEMIGROUPS

JONG MOON SHIN

ABSTRACT. This paper gives some basic properties on a disjoint decomposition of regular $*$–semigroups and shows that a regular $*$–semigroup with a left magnifying element has an identity element.

1. Introduction

In [6] a general disjoint decomposition of semigroups was given, which can be applied for the case of regular $*$–semigroups. In [4], [6] and related papers, authors obtained many properties of decomposition of regular semigroups relative to this decomposition. In this paper we discuss a decomposition of regular $*$–semigroups based on this decomposition. And we investigate the components of this decomposition and the interrelations between them. For two sets A and B we write $A \subset B$ if A is a proper subset of B.

2. Decomposition of semigroups

Let S be a semigroup without nonzero annihilator. Then S has the following disjoint decomposition;

$$(1) \qquad S = \bigcup_{i=0}^{5} S_i,$$

Received April 13, 1995. Revised June 3, 1995.
1991 AMS Subject Classification: 20-D99
Key words: regular $*$–semigroups, a magnifying element.

where

$$\begin{aligned} S_0 =&\{a \in S \mid aS \subset S \text{ and } \exists x \in S,\ x \neq 0 \text{ and } ax = 0\} \\ S_1 =&\{a \in S \mid aS = S \text{ and } \exists y \in S,\ y \neq 0 \text{ and } ay = 0\} \\ S_2 =&\{a \in S \setminus (S_0 \cup S_1) \mid aS \subset S \text{ and } \exists x_1, \\ & x_2 \in S,\ x_1 \neq x_2 \text{ and } ax_1 = ax_2\} \\ S_3 =&\{a \in S \setminus (S_0 \cup S_1) \mid aS = S \text{ and } \exists y_1, \\ & y_2 \in S,\ y_1 \neq y_2 \text{ and } ay_1 = ay_2\} \\ S_4 =&\{a \in S \setminus (S_0 \cup S_1 \cup S_2 \cup S_3) \mid aS \subset S\} \\ S_5 =&\{a \in S \setminus (S_0 \cup S_1 \cup S_2 \cup S_3) \mid aS = S\}. \end{aligned}$$

The subsets P_i $(i = 0, 1, \dots, 5)$ are defined analogously with multiplication by the element a on the right instead of the left. If the subsets $S_i(P_i)$ are nonempty then they are subsemigroups of S. Our theorems concern the decomposition (1), but analogous results can be formulated for the decomposition of $\{P_i\}$.

THEOREM 2.1. ([6]) *Let S be a semigroup.*

(1) *S_5 is a right group. Dually P_5 is a left group.*
(2) *$S_0 \cup S_2$ is a subsemigroup of S.*

DEFINITION. An element a of a semigroup S is said to be magnifying if there exists a proper subset M of S such that $aM = S$. Dually a right magnifying element is defined.

Note that any idempotent of S is not a left (right) magnifying element [3].

THEOREM 2.2. ([6]) *Let S be a semigroup. An element $a \in S$ is left magnifying*
if and only if $a \in S_1 \cup S_3$.

We recall that a semigroup S is regular if for every $a \in S$ there exists an $x \in S$ such that $a = axa$ and $x = xax$ (x is an inverse of a). The element ax and xa are idempotents. Since $aS = axaS \subseteq axS \subseteq aS$, $axS = aS$. Similarly, $xaS = xS$. The regular semigroup S can contain a zero element. Hence the components S_0 and S_1 can exist in the decomposition (1).

THEOREM 2.3. ([6]) *Let S be a regular semigroup. Then the inverses of the elements of $S_1 \cup S_3$ are in S_4 and the inverses of the elements of S_4 are in $S_1 \cup S_3$.*

3. Decomposition of regular $*-$Semigroups

DEFINITION. A semigroup S with a unary operation $* : S \longrightarrow S$ is called a $*-$semigroup if it satisfies
(i) $(x^*)^* = x$ for all $x \in S$,
(ii) $(xy)^* = y^*x^*$ for all $x,\ y \in S$.

A $*-$semigroup S is called a regular $*-$semigroup if $x = xx^*x$ for all $x \in S$.

THEOREM 3.1. *Let S be a regular semigroup. Then $S_0 \cup S_2$ is a regular semigroup.*

PROOF. From Theorem 2.1., $S_0 \cup S_2$ is a subsemigroup of S. Let $a \in S_0 \cup S_2$. Then $aS \subset S$ and $a = axa$. Assume that $xS = S$. Then $x(aS) = (xa)S = xS = S$. Thus x is a left magnifying element of S. This implies $x \in S_1 \cup S_3$. By Theorem 2.3., $a \in S_4$, which leads to a contradiction. We thus have $xS \subset S$ and so $x \in S_0 \cup S_2 \cup S_4$. Assume $x \in S_4$. Then $a \in S_1 \cup S_3$. This is a contradiction. So $x \in S_0 \cup S_2$. Hence $S_0 \cup S_2$ is a regular semigroup.

LEMMA 3.2. *Let S be a regular semigroup and let $x \in S$. Then $x \in S_5$ if and only if the inverses of x are in S_5.*

PROOF. Let $x \in S_5$ and $x = xyx$. Assume $y \in S \setminus S_5$. Then $y \in S_0 \cup S_2$ or $y \in S_1 \cup S_3$ or $y \in S_4$. This implies $x \in S_0 \cup S_2$ or $x \in S_4$ or $x \in S_1 \cup S_3$ by Theorems 2.1, 2.3 and 3.1. This is a contradiction. Hence $y \in S_5$. Similarly $y \in S_5$ implies $x \in S_5$.

THEOREM 3.3. *Let S be a regular $*-$semigroup.*
(1) *For any $x \in S$ and $i = 0, 1, \cdots, 5$, one has*
$x \in S_i$ *if and only if* $x^* \in P_i$.
(2) $S_1 \cup S_3 = P_4$ *and* $P_1 \cup P_3 = S_4$.
(3) $S_0 \cup S_2 = P_0 \cup P_2$.
(4) $S_5 = P_5$ *is a group.*

PROOF. (1) (i) Let $x \in S$ with $xy = 0$ for some $y \in S, y \neq 0$. Then $y^* \neq 0$ and $y^*x^* = (xy)^* = 0$.

(ii) Let $x \in S$ in which there exists $y, z \in S, y \neq z$ and $xy = xz$. Then $y^* \neq z^*$ and $y^*x^* = (xy)^* = (xz)^* = z^*x^*$.

(iii) Let $x \in S$ with $xS \subset S$. Assume that $Sx^* = S$. Then for any $a \in S$ there exists $b \in S$ such that $a^* = bx^*$. So $a = (a^*)^* = (bx^*)^* = xb^* \in xS$. Thus $S = xS$. This is a contradiction. Hence $xS \subset S$ implies $Sx^* \subset S$. Similarly $Sx^* \subset S$ implies $xS \subset S$.

(iv) Let $x \in S$ with $xS = S$. Then for any $a \in S$ there exists $b \in S$ such that $a^* = xb$. So $a = (a^*)^* = (xb)^* = b^*x^* \in Sx^*$. Thus $S = Sx^*$. Hence $xS = S$ implies $Sx^* = S$. Similarly $Sx^* = S$ implies $xS = S$. By combining these cases, we have $x \in S_i$ iff $x^* \in P_i, i = 0, 1, \ldots, 5$.

(2) Let $a \in S_1 \cup S_3$. Then by (1), $a^* \in P_1 \cup P_3$. So $a \in P_4$ by duality of Theorem 2.3. Conversely, let $a \in P_4$. Then $a^* \in S_4$ by (1). By theorem 2.3., $a \in S_1 \cup S_3$. Hence $S_1 \cup S_3 = P_4$. Dually we have $P_1 \cup P_3 = S_4$.

(4) By Lemma 3.2., $x \in S_5$ iff $x^* \in S_5$ iff $x = (x^*)^* \in P_5$ by (1). Thus $S_5 = P_5$. Since S_5 is a right group and P_5 is a left group, $S_5 = P_5$ is a group.

(3) Since $S_1 \cup S_3 \cup S_4 \cup S_5 = P_4 \cup P_1 \cup P_3 \cup P_5$, $S_0 \cup S_2 = P_0 \cup P_2$.

THEOREM 3.4. *If a regular $*-$semigroup S has a left magnifying element, then S has an identity element.*

PROOF. Let $a \in S$ be a left magnifying element of S. Then there exists a proper subset M of S such that $aM = S$, also $aS = S$. Let $y \in S$. Then there exists $x \in S$ such that $y = ax$. So $(aa^*)y = (aa^*)(ax) = (aa^*a)x = ax = y$. Thus aa^* is a left identity of S. If we put $aa^* = e$ then $e \notin S_1 \cup S_3$. Assume that $e \in S_0 \cup S_2$. Then there exists $x, y \in S, x \neq y$ such that $ex = ey$, where either x or y may be 0. Thus $x = y$. This is a contradiction. Thus $e \notin S_0 \cup S_2$. Assume $e \in S_4$. Then $e^* \in P_4$. This implies that $e \in S_1 \cup S_3$. This is a contradiction. So $e \notin S_4$. Hence $e \in S_5$. Since S is a regular $*-$semigroup, $S_5 = P_5$. Thus $eS = S = Se$. Hence e is an identity of S.

References

1. F. Catino and M. Gabriella Murciano,, *On some semigroups without increasing elements*, Note di Matematica **III** (1983), 1-13..

2. T. Imaoka, *Some remarks on fundamental regular $*-$semigroups*, LNM 998, Springer-Verlag (1981), 270-280.
3. E. S. Ljapin, *Semigroups*, Transl. Math. Monographs, Vol. 3 Providence R. I., 1963.
4. F. Lozzi and F. Miglinorini, *Γ-decomposition of regular semigroups and classification of bands*, Acta Math. Hung. **54 (3-4)** (1989), 209-223.
5. F. Migliorini and J. Szép, *On Γ-decomposition of semigroups*, LNM 998, Springer-Verlag (1981), 385-403.
6. F. Migliorini and J. Szép, *On a special decomposition of regular semigroups*, Acta. Sci. Math. **40** (1978), 121-128.
7. B. Piochi, *On Szép's decompositions of idempotent - generated semigroups*, Acta Math. Acad. Sci. Hungar. **40 (3-4)** (1982), 295-305 .
8. M. Petrich, *Introduction to semigroups*, C. E. Merril publ. Co. Columbus, Ohio, 1973.

Department of Mathematics
Dongguk University
Kyongju 780-714, Korea