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THETA SERIES BY PRIMITIVE ORDERS

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ABSTRACT. With the theory of a certain type of orders in a Quaternion algebra, we construct Brandt matrices and theta series. As a application, we calculate the class number of a certain type of orders in a Quaternion algebra with the trace formula of Brandt matrices.

1. Introduction

It is well known that there is a close connection between the theory of orders in Quaternion algebra and modular forms of $\Gamma_0(N)$ [2], [4]. There are three types of orders in Quaternion algebra (See Definition 2.1 below). Among them, two types of orders, so called, special orders were studied in [4]. The remaining type was studied in [1] and [6], in different ways. As a consequence of [6], in this paper we define theta series associated with a certain type of orders in a rational Quaternion algebra. With the results of [6], we obtain a trace formula for the Brandt matrices, which will play a central role in determining the subspace of cusp forms generated by the theta series (See [7]). For an immediate application of trace formula, we obtain an explicit formula for class number of primitive orders.

2. Primitive orders in Quaternion Algebra

Let Q be the rational number field and Z be the ring of integers in Q. For a prime p of Q, we denote as Q_p the completion of Q at p, and for $p < \infty$, denote as Z_p the ring of integers in Q_p. Let A be a Quaternion algebra over Q_p. A prime p is said to ramify in A if $A_p = A \otimes Q_p$ is

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a division algebra over Q_p (see [8; p154]). Otherwise A_p is isomorphic to Mat(2, Q_p) over Q_p and p is said to split in A (see [18; p184]). A lattice on A is a finitely generated Z submodule of A which contains a basis of A over Q. Since Z is a principal ideal domain, a lattice is a free Z module of rank 4. An order M of A is a lattice on A which is a subring containing the identity. There is a local-global correspondence for lattices which goes as follows [17; chapterIV]: to a lattice L on A, we associate the collection of lattices $L_p = L \otimes_Z Z_p$ of A_p, one for each $p < \infty$. Conversely, if we have a collection of lattices $\{L(p) | p < \infty\}$ on A_p, one for each $p < \infty$ and if there exists a lattice M on A such that $L(p) = M_p$ for almost all p, i.e. for all but a finite number of p, then there exists a unique lattice L on A such that $L(p) = L_p$ for all $p < \infty$. Replacing the word "lattice" by "order" above, we obtain the local-global correspondence for orders. An order of A(resp. A_p) is said to be maximal if it is not properly contained in any other order of A(resp. A_p) where p is a finite prime.

DEFINITION 2.1. An order M of A is said to be primitive if

1) for all finite ramified primes p of A, M_p contains a subring which is Z_p isomorphic to the ring of integers in some quadratic field extension of Q_p.
2) for all finite split primes p of A, M_p contains a subring which is Z_p isomorphic to the ring of integers in some quadratic field extension of Q_p or isomorphic to $Z_p \oplus Z_p$ in quadratic extension $Q_p \oplus Q_p$ of Q_p.

REMARK. For all ramified primes p of A, pritimitive orders M of A were studied by Hijikata, Pizer and Shemanske [4]. Also, for all finite split primes p of A, orders M_p of A_p which contain a subring which is Z_p-isomorphic to $Z_p \oplus Z_p$ were studied by Hijikata [3].

2.1 Now let us restrict to the case that really interests us at present. For the remainder of this paper, A will be a rational Quaternion algebra ramified precisely at one finite prime q and ∞. Thus $A_\infty = A \otimes_Q R$ is Hamilton's Quaternion algebra [10; p343].

If R is an order of A_p which contains $\mathcal{O}_L$, the ring of integers in a quadratic field extension L of Q_p for $p \neq q$, then the possiblities for R

are:

$$R = \begin{cases} R_{2\nu}(L) = \mathcal{O}_L + \xi P_L^{\nu} \text{ if } L \text{ is unramified} \\ R_{\nu}(L) = \mathcal{O}_L + (1+\xi)P_L^{\nu-1} \text{ if } L \text{ is ramified} \\ \bar{R}_0(L) = \mathcal{O}_L + (1-\xi)P_L^{-1} \text{ if } L \text{ is ramified} \end{cases}$$

for some nonnegative integer ν where $A_p = L + \xi L$ and P_L is the prime ideal of $\mathcal{O}_L$(See [6]).

DEFINITION 2.2. Let A be a rational Quaternion algebra which is ramified precisely at one finite prime q and ∞. For finite odd primes $p_1, p_2, ..., p_d \neq q$, an order of M of A is said to have level $(q; L(p_1), \nu(p_1); L(p_2), \nu(p_2); ..., L(p_d), \nu(p_d))$ if

i) M_q is the maximal order of A_q.

ii) for a prime $p \neq q$, there exists a quadratic field extension $L(p)$ of Q_p and a nonnegative integer $\nu(p)$ (which is even if $L(p)$ is unramified) such that $M_p = R_{\nu(p)}(L(p))$

iii) $\nu(p_i) > 0$ for $i = 1, 2, ..., d$ and $\nu(p) = 0$ for $p \neq q, p_1, ..., p_d$. (i.e. M_p is a maximal order of A_p if $p \neq p_1, p_2, ..., p_d$).

REMARK. For notational convenience, we put $N' = (q; L(p_1), \nu(p_1); ..., L(p_d), \nu(p_d))$ and $N = q\prod_{i=1}^{d} p_i^{\nu(p_i)}$ throughout this paper.

DEFINITION 2.3. Let M be an order of level N' in A. A left M ideal I is a lattice on A such that $I_p = M_p a_p$ for some $(a_p \in A_p^{\times})$ for all $p < \infty$. Two left M ideals I and J are said to belong to the same class if $I = Ja$ for some $a \in A^{\times}$. One has the obvious analogous definitions for right M ideals.

DEFINITION 2.4. The class number of left ideals for any order M of level
$N' = (q; L(p_1), \nu(p_1); ..., L(p_d), \nu(p_d))$ is the number of distinct classes of such ideals. We denote this class number by $H(N')$.

DEFINITION 2.5. Let I be a (left or right) M ideal for some order M of level N' in A. The left order of $I = \{a \in A | aI \subset I\}$ and the right order of $I = \{a \in A | Ia \in I\}$.

DEFINITION 2.6. The norm of an ideal, denoted by $N(I)$, is the positive rational number which generates the fractional ideal of Q generated by $\{N(a)|a \in I\}$. The conjugate of an ideal I, denoted by $\bar{I}$, is given by $\bar{I} = \{\bar{a}|a \in I\}$. The inverse on an ideal, denoted by I^{-1}, is given by $I^{-1} = \{a \in A|IaI \subset I\}$.

REMARK. Locally, if $I_p = M_p a_p$ for some $a_p \in A_p^\times$, then we define $N(I_p) = N(a_p) \mod Z_p^\times$.

Note: If we have two ideals I and J with right order of I equal to the left order of J, then IJ (= all finite sums $\sum_{i_k \in I, j_k \in J}(i_k j_k)$ with $i_k \in I$ and $j_k \in J$) is an ideal with left order equal to the left order of I and right order equal to right order of J (see [16; p210]).

PROPOSITION 2.7. *Let M be an order of level $N' = (q; L(p_1), \nu(p_1); ..., L(p_d), \nu(p_d))$. Let I be a left M ideal with right order M'. Then*

i) *$\bar{I}$ is a left M' ideal with right order M and $N(\bar{I}) = N(I)$.*

ii) *$II^{-1} = M$ and $I^{-1}I = M'$.*

iii) *I^{-1} is a left M' ideal with right order M and $N(I^{-1}) = N(I)^{-1}$.*

PROOF. i) By Definition 2.5, it is clear that $\bar{I}$ is a Z lattice. Furthermore,

$$\begin{aligned}(\bar{I})_p &= \bar{I} \otimes Z_p = \overline{I_p} = \overline{M_p a_p} \text{ for some } a_p \in A_p \\ &= \overline{a_p} M_p = (\overline{a_p} M_p \overline{a_p}^{-1})\overline{a_p} = M'_p \overline{a_p}\ .\end{aligned}$$

Therefore, $\bar{I}$ is a left M' ideal with right order M. $N(\bar{I}) = N(I)$ follows from $\{N(\bar{a})|a \in I\} = \{N(a)|a \in I\}$.

ii) The proofs that $II^{-1} = M$ and $I^{-1}I = M'$ are given in [16; p192 Theorem 22.7].

iii) $I^{-1} = \{a \in A|IaI \subset I\} = \{x \in A|Ix \subset M\}$ (See [16; p192 (22.6)]). By Definition 2.3, $I_p = M_p a_p$ for some $a_p \in A_p$ for each $p < \infty$. Therefore, $(I_p)^{-1} = \{x \in A_p|M_p a_p x \subset M_p\} = a_p^{-1} M_p$, which implies $I_p^{-1} = M'_p a_p^{-1}$ for all $p < \infty$. Thus we have proven that I^{-1} is a left M' ideal with right order M.

For the proof of $N(I^{-1}) = N(I)^{-1}$, see Theorem 24.5 [16; p212]. This completes the proof.

PROPOSITION 2.8. [Pizer] *Let M be an order of level N' in A. Let $I_1, I_2, ..., I_H$ be a complete set of representatives of all the distinct left M ideal classes. Let M_j be the right order of $I_j, j = 1, 2, ..., H$. Then $I_j^{-1} I_1, ..., I_j^{-1} I_H$ is a complete set of representatives of all distinct left M_j ideal classes (for $i = 1, 2, ..., H$).*

PROOF. See Proposition 2.13 and Proposition 2.15 [13].

3. Brandt matrices and Theta series

3.1 We now give the connection between modular forms and Quaternion algebras. Let $Q(x)$ be a positive definite integral quadratic form in an even number of $r = 2k$ variables. Integral means that $Q(x) \in Z$ for all $x \in Z^r$. Then $Q(x) = \frac{1}{2}x^t T x$ where $x^t = (x_1, x_2, ...x_r)$ and $T = (a_{ij})$ is a positive definite symmetric matrix with $a_{ij} \in Z$ and $a_{ii} \equiv 0 \mod 2$. In fact, T is the matrix of the bilinear form $(x, y) = Q(x+y) - Q(x) - Q(y)$. T is called the matrix associated to $Q(x)$.

DEFINITION 3.1. Let $Q(x)$ and T be as above. The level of $Q(x)$ T) is the least positive integer n such that nT^{-1} has integer entries with diagonal entries even integers. The discriminant of $Q(x)$ is $(-1)^k \det(T)$.

PROPOSITION 3.2. *Let I be a left M ideal for some order M of level $N' = (q; L(p_1), \nu(p_1); ..., L(p_d), \nu(p_d))$ in a positive definite Quaternion algebra A over Q which is ramified precisely at one finite prime q and ∞. Then the quadratic form $N(x)/N(I)$ for $x \in I$ is a positive definite integral quadratic form with level N and discriminant N^2 where $N = q\prod_{i=1}^{d} p_i^{\nu(p_i)}$.*

REMARK. What this means is the following. Let $e_1, \cdots, e_4$ be any Z basis for I. Then $Q(x_1, \cdots, x_4) = N(x_1 e_1 + \cdots + x_4 e_4)/N(I)$ is a positive definite integral quadratic form with level N and discriminant N^2. Since any other Z-basis of I is obtained from $e_1, \cdots, e_4$ by operating on $(e_1, \cdots, e_4)$ by a matrix $U \in GL(4, Z) = \{S \in \mathrm{Mat}_{4\times 4}(Z) | \det(S) = \pm 1\}$, the level and the discriminant are independent of which particular basis we chose.

PROOF. Let $Q(x) = N(x)/N(I)$. Since $A_\infty = A \otimes R$ is Hamilton's Quarternion, the norm form is positive definite by [11; p343]. Hence $Q(x)$ is a positive definite form. Next, by the Definition 2.6, $N(I)|N(x)$ for all $x \in I$. This implies $Q(x) = N(x)/N(I)$ is integral.

We now need to show that $Q(x)$ has level N and discriminant N^2. Let S be the matrix associated to $Q(x)$. As the level is a positive integer, we determine the level locally at all primes $< \infty$.

We start to consider the case $p \neq q$ first. By Definition 2.3, $I_p = M_p\beta$ for some $\beta \in A_p^\times$. By 2.1, $M_p = R_{\nu(p)}(L(p))$ for some nonnegative integer $\nu(p)$. Suppose e_1, e_2, e_3, e_4 is a basis of R_ν. Then $e_1\beta, e_2\beta, e_3\beta, e_4\beta$ gives a Z_p basis for I_p. Since $N(I_p) = N(\beta)$ (see Remark of Definition 2.6),

$$\begin{aligned} \text{the} \quad ij\text{-th entry of } S \text{ is} \quad & Q(e_i\beta + e_j\beta) - Q(e_i\beta) - Q(e_j\beta) \\ &= \frac{1}{N(I_p)}(N(\beta)(N(e_i + e_j) - N(e_i) - N(e_j))) \\ &\equiv N(e_i + e_j) - N(e_i) - N(e_j) = Tr(e_i\bar{e_j}) \mod Z_p^\times . \end{aligned}$$

First consider the case, $\nu(p) > 0$. Let $\nu = \begin{cases} \frac{\nu(p)}{2} & \text{if } L(p) \text{ is unramified} \\ \nu(p) - 1 & \text{if } L(p) \text{ is ramified} \end{cases}$,

and $L = L(p)$. Then $R_\nu = \mathcal{O}_L + \xi P_L^\nu$. Let $\mathcal{O}_L = Z_p + uZ_p$ for some u in L, so that $\mathcal{O}_L$ is the ring of integers in L. Now we take $e_1 = 1, e_2 = u, e_3 = \xi\pi_L^\nu, e_4 = \xi\pi_L^\nu u$ as a Z_p basis of $M_p = R_\nu(L)$. Since $\overline{\xi\pi_L^\nu} = -\xi\pi_L^\nu$ and $\overline{\xi\pi_L^\nu u} = -\xi\pi_L^\nu u$ where π_L is the prime element of $\mathcal{O}_L$ (See [6]),

$$S = \begin{pmatrix} 2 & \mathrm{Tr}(u) & 0 & 0 \\ \mathrm{Tr}(u) & 2\mathrm{N}(u) & 0 & 0 \\ 0 & 0 & 2\mathrm{N}(\pi_L^\nu) & -\mathrm{N}(\pi_L^\nu)\mathrm{Tr}(u) \\ 0 & 0 & -\mathrm{N}(\pi_L^\nu)\mathrm{Tr}(u) & 2\mathrm{N}(\pi_L^\nu u) \end{pmatrix}.$$

Let $\delta = 4\mathrm{N}(u) - \mathrm{Tr}(u)^2$. Then

$$S^{-1} = \begin{pmatrix} 2\mathrm{N}(u)/\delta & -\mathrm{Tr}(u)/\delta & 0 & 0 \\ -\mathrm{Tr}(u)/\delta & 2/\delta & 0 & 0 \\ 0 & 0 & 2\mathrm{N}(\pi_L^\nu)\mathrm{N}(u)/\delta\mathrm{N}(\pi_L^\nu)^2 & \mathrm{N}(\pi_L^\nu)\mathrm{Tr}(u)/\delta\mathrm{N}(\pi_L^\nu)^2 \\ 0 & 0 & \mathrm{N}(\pi_L^\nu)\mathrm{Tr}(u)/\delta\mathrm{N}(\pi_L^\nu)^2 & 2\mathrm{N}(\pi_L^\nu)/\delta\mathrm{N}(\pi_L^\nu)^2 \end{pmatrix}$$

so the level and the discriminant of $Q(x) = \frac{\mathrm{N}(x)}{N(I)}$ are $(4\mathrm{N}(u) - \mathrm{Tr}(u)^2)\mathrm{N}(\pi_L^{\nu}) \mod Z_p^{\times}$ and $(4\mathrm{N}(u) - \mathrm{Tr}(u)^2)^2\mathrm{N}(\pi_L^{\nu})^2 \mod Z_p^{\times}$, respectively.

If $L(p)$ is an unramified extension field of Q_p, then $\nu = \frac{\nu(p)}{2}$ and $\Delta(u)$ is a quadratic nonresidue mod p in Q_p, whence $\Delta(u) = -(4\mathrm{N}(u) - \mathrm{Tr}(u)^2)$ is a unit in Z_p. On the other hand, if $L(p)$ is a ramified extension field of Q_p, then $\nu = \nu(p) - 1$ and $u = \pi_L$. Hence $\Delta(\pi_L) = -(4N(\pi_L) - Tr(\pi_L)^2) \equiv p \mod Z_p^{\times}$.

In both cases, the level of $Q(x) \mod Z_p^{\times} = p^{\nu(p)}$ The discriminant of $Q(x) = \frac{N(x)}{N(I)}$ mod units of Z_p is $\mathrm{disc}(M_p) = \det(Tr(e_i\bar{e_j})) = \det(S) = (4\mathrm{N}(u) - \mathrm{Tr}(u)^2)^2\mathrm{N}(\pi_L^{\nu})^2$. That is, the discriminant of $Q(x) \mod Z_p^{\times} = p^{2\nu(p)}$ Thus the level and the discriminant of $Q(x)$ mod units of Z_p are $p^{\nu(p)}$ and $p^{2\nu(p)}$ respectively.

If $\nu(p) = 0$, M_p is a maximal order of A_p, in which case the level and discriminant of $\frac{N(x)}{N(I)}$ are both 1 mod units of Z_p (see [14 ; Proposition 2.11]).

In the case, $p = q$, the level and discriminant of $\frac{N(x)}{N(I)}$ mod units of Z_p, q and q^2, have been calculated by A. Pizer[14] and [19].

We conclude that the discriminant of $Q(x)$ is $q^2 \prod_{p|p_1p_2\cdots p_d} p^{2\nu(p)}$ and the level of $Q(x)$ is $q\prod_{p|p_1p_2\cdots p_d} p^{\nu(p)}$.

This completes the proof.

3.2 Let M be an order of level $N' = (q; L(p_1), \nu(p_1); ..., L(p_d), \nu(p_d))$ in a Quaternion algebra A over Q ramified precisely at one finite prime q and ∞. Let $I_1, I_2, ..., I_H$, $H = H(N')$ be representatives of all distinct left M ideal classes. Let M_j be the right order of I_j and $e_j = |U(M_j)|$. We define

$$b_{ij}(n) = \frac{1}{e_j} \sum_{\alpha \in I_j^{-1}I_i, N(\alpha) = nN(I_i)/N(I_j)} 1 \quad \text{and} \quad b_{ij}(0) = \frac{1}{e_j}.$$

Then $b_{ij}(n) = \frac{1}{e_j}\cdot$(the number of elements in $I_j^{-1}I_i$ whose norms are $nN(I_i)/N(I_j)$ for $n > 0$).

We are now in position to define the Brandt matrices associated with the primitive orders in Quaternion algebra.

DEFINITION 3.3. Let the notation be as above. The Brandt matrices for $n \geq 0$ are defined by

$$B(n : N') = (b_{ij}(n)) .$$

Thus $B(n : N')$ is an $H \times H$ matrix with $b_{ij}(n)$ as the ij-th entry.

THEOREM 3.4. *The entries of the Brandt matrix series,*

$$\Theta(\tau : N') = \sum_{n=0}^{\infty} B(n : N') e^{2\pi i n \tau}$$

are modular forms of weight 2 on $\Gamma_0(N)$.

PROOF. Recall that $B(n : N') = (b_{ij}(n))$ where $b_{ij}(n)$ is just $\frac{1}{e_j}$ times the number of elements $\alpha \in I_j^{-1} I_i$ with $N(\alpha) = nN(I_i)/N(I_j)$ for $n > 0$. Each entry of the Brandt matrix series, $\Theta(\tau : N') = (\theta_{ij}(\tau))$, is

$$\begin{aligned} \theta_{ij}(\tau) &= \sum_{n=0}^{\infty} b_{ij}(n) e^{2\pi i n \tau} \\ &= \frac{1}{e_j} \sum_{x \in I_j^{-1} I_i, N(x) = nN(I_i)/N(I_j)} e^{2\pi i n \tau} \\ &= \frac{1}{e_j} \sum_{x \in I_j^{-1} I_i} e^{2\pi i \tau N(x) N(I_j)/N(I_i)} . \end{aligned}$$

Let $Q(x) = N(x)N(I_j)/N(I_i)$. Since $I_j^{-1} I_i$ is a left ideal of M_j, it is a free Z module of rank 4. So identifying $I_j^{-1} I_i$ with Z^4, we have $\theta_{ij}(\tau) = \frac{1}{e_j} \sum_{x \in Z^4} e^{2\pi i \tau Q(x)}$. By Theorem 20 of [9: VI22] and Proposition 3.2 above, this is a modular form of weight 2 on $\Gamma_0(N)$. Note that the spherical function with respect to $Q(x)$ is 1 in the notation of Ogg [9: VI22] and the character associated to $\theta_{ij}(\tau)$ is 1, since by Proposition 6.12 disc$(Q(x)) = N^2$ and Theorem 20 of [9: VI22] shows that $\epsilon(d) = (\frac{N^2}{d}) = 1$. This completes the proof.

Our final goal is to find the trace formula for the Brandt matrix $B(n : N')$, which will be the central role in determining the subspace of modular forms generated by theta series (See [7]). First we need to determine the mass formula for M ideals. Let M be an order of level $N' = (q; L(p_1), \nu(p_1); \dots, L(p_d), \nu(p_d))$ on A and $I_1, I_2, \dots, I_H$ be representatives of the left M ideal classes. Recall that the right order of I_i is given by $M_i = \{a \in A | I_i a \subset I_i\}$.

DEFINITION 3.5. Let the notations be as above. The mass formula for M ideals where M is an order of level $N' = (q; L(p_1), \nu(p_1); \dots, L(p_d), \nu(p_d))$ is given by

$$\mathrm{Mass}(M) = 2\sum_{i=1}^{H} \frac{1}{|U(M_i)|} .$$

THEOREM 3.6. *Let M be an order of level $N' = (q; L(p_1), \nu(p_1); \dots, L(p_d), \nu(p_d))$ on A. Then*

$$Mass(M) = \frac{1}{12}(q-1)\prod_{i=1}^{d} \delta(p_i)$$

where $$\delta(p_i) = \begin{cases} (p_i^2 - p_i)p_i^{\nu(p_i)-2} & \text{if } L(p_i) \text{ is unramified} \\ (p_i^2 - 1)p_i^{\nu(p)-2} & \text{if } L(p_i) \text{ is ramified and } \nu(p_i) \geq 2 \\ (p_i + 1) & \text{if } L(p_i) \text{ is ramified and } \nu(p_i) = 1 . \end{cases}$$

PROOF. Let M^0 be an order of level q in A which contains M. Then as in Proposition 24 and Proposition 25 [12; p685],

$$\mathrm{Mass}(M) = \mathrm{Mass}(M^0)([U(M^0) : U(M)]) .$$

By Eichler[2; p95] $\mathrm{Mass}(M^0) = \frac{1}{12}(q-1)$. Thus we need to find $[U(M^0) : U(M)]$.

By Corollary1 [18; p88],

$$[U(M^0) : U(M)] = \prod_p [U(M_p^0) : U(M_p)] .$$

Since M_p^0 is a maximal order, $M_p^0 = R_0(L(p))$ and $M_p = R_{\nu(p)}(L(p))$.

Suppose $p \neq p_1, \cdots, p_d$. Then $M_p^0 = M_p$, which implies $[M_p^0 : M_p] = 1$. Hence we consider $p = p_i$ for some $1 \leq i \leq d$. In the following calculations, $[R_i^\times : R_{i+1}^\times]$ is given in Proposition 2.4 and Proposition 2.7 [6]. If $L(p)$ is unramified over Q_p, then

$$\begin{aligned}[U(M_p^0) : U(M_p)] &= [R_0^\times : R_2^\times] \cdots [R_{\nu(p)-2}^\times : R_{\nu(p)}^\times] \\ &= (p^2 - p)p^2 \cdots p^2 \\ &= (p^2 - p)p^{\nu(p)-2} .\end{aligned}$$

If $L(p)$ is ramified over Q_p and $\nu(p) \geq 2$, then

$$\begin{aligned}[U(M_p^0) : U(M_p)] &= [R_0^\times : R_1^\times][R_1^\times : R_2^\times] \cdots [R_{\nu(p)-1}^\times : R_{\nu(p)}^\times] \\ &= (p+1)(p-1)pp \cdots p \\ &= (p^2 - 1)p^{\nu(p)-2} .\end{aligned}$$

Finally, if $L(p)$ is ramified over Q_p and $\nu(p) = 1$, then

$$[U(M_p^0) : U(M_p)] = [R_0^\times : R_1^\times] = p + 1 .$$

Hence

$$\text{Mass}(M) = \frac{1}{12}(q-1)\prod_{i=1}^{d} \delta(p_i).$$

This completes the proof.

3.3 We need to set some notations. Let K be an imaginary quadratic number field and $\mathcal{O}$ an order of K. Let A be a Quaternion algebra over Q ramified only at q and ∞ and M an order of level N' of A.

Analogously as in the local case, an optimal embedding $\mathcal{O}/K$ into M/A is an Q injective homomorphism φ, such that $\varphi(K) \cap M = \varphi(\mathcal{O})$. Then we denote by $A(\mathcal{O}, M)$, the number of mod $U(M)$ equivalence classes of optimal embeddings of $\mathcal{O}/K$ into M/A. Note that $A(\mathcal{O}, M)$ depends only on the isomorphism classes of $\mathcal{O}$ and M. For a prime l, denote by $C_l(\mathcal{O})$ the number of mod $U(M_l)$ equivalence classes of optimal embedding of $\mathcal{O}_l/K_l$ into M_l/A_l (See 5.2 and Definition 5.1 in [6]). Note that $C_l(\mathcal{O})$ depends only on $\mathcal{O}_l$ and the level of M_l.

Let M be an order of level $N' = (q; L(p_1), \nu(p_1); ..., L(p_d), \nu(p_d))$ of A. Let $I_1, I_2, .., I_H$ be a set of representatives of all the left M ideal classes and M_j be the right order of I_j for $1 \leq j \leq H$.

THEOREM 3.6. [Pizer] *Let the notation be as above. Then we have*

$$\sum_{i=1}^{H} A(\mathcal{O}, M_i) = h(\mathcal{O}) \prod_{l|N} C_l(\mathcal{O}) .$$

where $h(\mathcal{O})$ is the class number of locally principal $\mathcal{O}$ ideals and the product is over all primes l dividing N.

PROOF. See Theorem 4.8 [15; p192].

COROLLARY 3.7. [Pizer] *In the notation of 3.3, let $a_i(\mathcal{O})$ denote the number of optimal embeddings of $\mathcal{O}/K$ into M_i/A. Then*

$$\sum_{i=1}^{H} \frac{a_i(\mathcal{O})}{e_i} = \frac{h(\mathcal{O})}{|U(\mathcal{O})|} \prod_{l|N} C_l(\mathcal{O})$$

where $e_i = |U(M_i)|$.

PROOF. See Corollary 4.10 [15; p192].

THEOREM 3.8. *The trace of Brandt matrix $B(n : N')$ is*

$$tr(B(n : N')) = \sum_{s} \sum_{f} \frac{1}{2} b(s, f) \prod_{l|N} c(s, f, l) + \xi(\sqrt{n}) Mass(M)$$

where $\xi(\sqrt{n}) = \begin{cases} 1 & \text{if } n \text{ is a perfect square} \\ 0 & \text{otherwise} . \end{cases}$

The meaning of $s, f, b(s, f)$ and $c(s, f, l)$ are as follows.

Let s run over all integers such that $s^2 - 4n$ is negative. Hence with some positive integer t and square free integer m, we can classify $s^2 - 4n$ by

$$s^2 - 4n = \begin{cases} t^2 m & m \equiv 1 \mod 4 \\ t^2 4m & m \equiv 2, 3 \mod 4 . \end{cases}$$

For each s, let f run over all positive divisors of t. Let $L = Q[x]/(\Phi_s(x))$ where $\Phi_s(x) = x^2 - sx + n$ and ξ is the canonical image of x in L. Then L

is an imaginary quadratic number field and ξ generates the order $Z+Z\xi$ of L. For each f, there is a uniquely determined order $\mathcal{O}_f$ containing $Z+Z\xi$ as a submodule of index f. Let $\Delta(\mathcal{O}_f)=s^2-4n/f^2$. Let $h(\Delta(\mathcal{O}_f))$(resp. $\omega(\Delta(\mathcal{O}_f))$ denote the number of locally principal $\mathcal{O}_f$ ideals(resp. $\frac{1}{2}|U(\mathcal{O}_f)|$). Then $b(s,f)=\frac{h(\Delta(\mathcal{O}_f))}{\omega(\Delta(\mathcal{O}_f))}$.

Let M be an order of level N' of B. Then $c(s,f,l)$ is the number of $M_l^\times=(M\otimes Z_l)^\times$ equivalence classes of optimal embeddings of $\mathcal{O}_f\otimes Z_l$ into $M\otimes Z_l$. In other words, let $Z+Z\alpha$ be the maximal order of L, then $\mathcal{O}_f\otimes Z_l=Z_l+Z_l l^m\alpha$ and $(s^2-4n)/f^2\equiv l^{2m}\Delta(\alpha)\mod (Z_l^\times)^2$. Since $c(s,f,l)$ is the number of $M_l^\times=R^\times_{\nu(l)}(L(l))$ (See 3.3) equivalence classes of optimal embeddings of $l^m\alpha$ into $M_l=R_{\nu(l)}(L(l))$, it is easy to find $c(s,f,l)$ in Theorem 5.19, 5.30, 5.31 and Table 5.28 in [6] or [1] if s,n and f are given.

REMARK. $h(\Delta(\mathcal{O}_f))$ can be expressed in terms of 'standard' class number of maximal orders (see Corollary 3.11). It is well known that $w(\Delta(\mathcal{O}_f))=1$ with two exceptions, $w(-4)=2$ and $w(-3)=3$ (see [19; p267]).

PROOF. Recall that $B(n:N')=(b_{ij}(n))$
where $b_{ij}(n)=\frac{1}{e_j}\sum_{\alpha\in I_j^{-1}I_i,N(\alpha)=nN(I_i)/N(I_j)}1$. Then

$$\begin{aligned}\mathrm{tr}B(n:N')&=\sum_{i=1}^{H}b_{ii}(n)\\&=\sum_{i=1}^{H}\frac{1}{e_i}\sum_{\alpha\in I_i^{-1}I_i,N(\alpha)=nN(I_i)/N(I_i)}1\\&=\sum_{i=1}^{H}\frac{1}{e_i}\sum_{\alpha\in M_i,N(\alpha)=n}1\,.\end{aligned}$$

If n is a perfect square, then $n=a^2$ for some $a\in Z$. Since M_i contains Z for each i and $N(\pm a)=a^2=n$, then $\sum_{\alpha\in M_i,N(\alpha)=n}1=2$ for each $1\le i\le H$. Hence

$$\sum_{i=1}^{H}\frac{1}{e_i}\sum_{\alpha\in M_i,N(\alpha)=n}1=2\sum_{i=1}^{H}\frac{1}{e_i}=\mathrm{Mass}(M)\,.$$

Now if n is not a perfect square in Q, then let $a_i(s,n)$ denote the number of $\alpha \in M_i$ with $\mathrm{tr}(\alpha) = s$, $\mathrm{N}(\alpha) = n$, and with $x^2 - sx + n$ irreducible over Q. Then $\sum_{\alpha \in M_i, \mathrm{N}(\alpha)=n} 1 = \sum_s a_i(s,n)$ where the sum is over all integers, s such that $s^2 - 4n < 0$.

$$\sum_{i=1}^{H} \frac{1}{e_i} \sum_s a_i(s,n) = \sum_{i=1}^{H} \sum_s \frac{a_i(s,n)}{e_i}$$
$$= \sum_s \sum_{i=1}^{H} \frac{a_i(s,n)}{e_i} .$$

Let $K = Q[x]/(x^2 - sx + n)$ and let x' be a root of $x^2 - sx + n$ in K. Then $a_i(s,n)$ is equal to the number of isomorphisms ϕ of K into A with $\phi(x') \in M_i$. Let $\mathcal{O}_0 = Z + Zx'$ and $\mathcal{O}_1$ be an order of K with $\mathcal{O}_0 \subset \mathcal{O}_1 \subset K$. If ϕ is an optimal embedding of $\mathcal{O}_1/K$ into M_i/A, then $\phi(\mathcal{O}_1) = M_i \cap \phi(K)$ and $x \in \mathcal{O}_0 \subset \mathcal{O}_1$ imply $\phi(x') \in M_i$. Thus every optimal embedding of some order $\mathcal{O}_1, \mathcal{O}_0 \subset \mathcal{O}_1 \subset K$ into M_i/A is an isomorphism which is counted in $a_i(s,n)$. Conversely, if $\phi : K \to A$ is an isomorphism with $\phi(x') \in M_i$ then $M_i \cap \phi(K) = \mathcal{O}_1'$ is an order of $\phi(K)$ containing $\phi(x')$. Hence $\phi^{-1}(\mathcal{O}_1')$ is an order of K which contains $\mathcal{O}_0$ and such that ϕ gives an optimal embedding of $\phi^{-1}(\mathcal{O}_1')$ into M_i. Thus $a_i(s,n) = \sum_{\mathcal{O}_1 \supset \mathcal{O}_0} a_i(\mathcal{O}_1)$, which we sum over all orders $\mathcal{O}_1$ of K which contain $\mathcal{O}_0$, and $a_i(\mathcal{O}_1)$ is as in Corollay 3.7. Hence we have

$$\sum_{i=1}^{H} \frac{a_i(s,n)}{e_i} = \sum_{\mathcal{O}_1 \supset \mathcal{O}_0} \sum_{i=1}^{H} \frac{a_i(\mathcal{O}_1)}{e_i}$$
$$= \sum_{\mathcal{O}_1 \supset \mathcal{O}_0} \frac{h(\mathcal{O}_1)}{|U(\mathcal{O}_1)|} \prod_{l|N} c_l(\mathcal{O}_1) .$$

by Corollary 3.7.

Now $\Delta(\mathcal{O}_0) = s^2 - 4n$ and $\Delta(\mathcal{O}_1) = (s^2 - 4n)/f^2$ where $(s^2-4n)/f^2 \equiv 0$ or $1 \mod 4$ and f is a positive integer. Taking into account the fact that K must be imaginary quadratic and that an order of K is uniquely determined by its discriminant, we set $h(\Delta(\mathcal{O}_1)) = h(\mathcal{O}_1), \omega(\Delta(\mathcal{O}_1)) =$

$\frac{1}{2}|U(\mathcal{O}_1)|$ and $c(s,f,l) = c_l(\mathcal{O}_1)$. Then

$$\sum_s \sum_{i=1}^{H} \frac{a_i(s,n)}{e_i} = \sum_s \sum_{\mathcal{O}_1 \supset \mathcal{O}_0} \frac{h(\mathcal{O}_1)}{|U(\mathcal{O}_1)|} \prod_{l|N} c_l(\mathcal{O}_1)$$
$$= \sum_s \sum_f \frac{1}{2} b(s,f) \prod_{l|N} c(s,f,l)\,.$$

Therefore,

$$\mathrm{tr}(B(n:N')) = \sum_s \sum_f \frac{1}{2} b(s,f) \prod_{l|N} c(s,f,l) + \xi(\sqrt{n})\mathrm{Mass}(M)$$

LEMMA 3.9. *Let K be an imaginary quadratic number field. Let $\mathcal{O}_K$ be an order of K of discriminant Δ and let $\mathcal{O}'$ be the suborder of $\mathcal{O}_K$ of index f. Then*

$$\frac{h(\mathcal{O}'_K)}{\omega(\mathcal{O}'_K)} = \frac{h(\mathcal{O}_K)}{\omega(\mathcal{O}_K)} f \prod_{l|f} (1 - \{\frac{\Delta}{l}\}\frac{1}{l})$$

where $\{\frac{\Delta}{l}\} = \begin{cases} 0 & \text{if } l^2|\Delta \text{ and } l^{-2}\Delta \equiv 0 \text{ or } 1 \mod 4 \\ (\frac{\Delta}{l}) & \text{the Kronecker symbol otherwise} \end{cases}.$

PROOF. See Lemma 4.16 [15; p197]

COROLLARY 3.10. *Let K be an imaginary quadratic number field. Let $\mathcal{O}$ be the maximal order of K and O' a suborder of index f. Then*

$$\frac{h(\mathcal{O}'_K)}{\omega(\mathcal{O}'_K)} = \frac{h(\mathcal{O}_K)}{\omega(\mathcal{O}_K)} f \prod_{l|f} (1 - (\frac{K}{l})\frac{1}{l})$$

where

$$(\frac{K}{l}) = \begin{cases} 1 & \text{if } l \text{ splits in } K \\ 0 & \text{if } l \text{ ramifies in } K \\ -1 & \text{if } l \text{ remains prime in } K \end{cases}.$$

is the Kronecker symbol. Note that $h(\mathcal{O}_K)$ is the standard class number of K.

PROOF. See Corollary 4.17 [15; p197].

3.4 Let L and L' be two quadratic extensions of Q_p contained in A_p. By an embedding we mean an injective Q_p (or Z_p) homomorphism.

Assume that $L \subset B$ and let $\mathcal{O}'$ be an order of L'. We say that $\mathcal{O}'$ is embeddable in $R_\nu(L)$ if there exists an embedding ϕ of L' into B such that $\phi(\mathcal{O}') \subset R_\nu(L)$.

DEFINITION 3.11. Define $\mu(L, L')$ to be the nonegative integer or ∞ characterized by the property : $\mathcal{O}_{L'}$ is embeddable in $R_\nu(L)$ if and only if $\nu \leq \mu(L, L')$.

Obviously, $\mu(L, L')$ exists and depends only on discriminants of L and L'. Also if discriminants of L and L' are equal, then $\mu(L, L') = \mu(L', L) = \infty$. For the details, see [6].

THEOREM 3.12. *Let A be a rational Quaternion algebra ramified precisely at one finite prime q and ∞ and M be an order of A of level $N' = (q; L(p_1), \nu(p_1); ..., L(p_d), \nu(p_d))$ where $2 \nmid \prod_{i=1}^d p_i$. Then the class number of an order M is*

$$H(N') = Mass(M) + \frac{1}{4}(1 - (\frac{-4}{q}))\prod_{l|\frac{N}{q}} C(l) + \frac{1}{3}(1 - (\frac{-3}{q}))\prod_{l|\frac{N}{q}} C'(l)\,,$$

where $N = q\prod_{i=1}^d p_i^{\nu(p_i)}$,

$$C(l) = \begin{cases} 2 & \text{if } \mu(Q_l(\sqrt{-1}), L(l)) = 1 \quad \text{and } \nu(l) = 1 \\ 2 & \text{if } \mu(Q_l(\sqrt{-1}), L(l)) = \infty \\ 0 & \text{otherwise}\,, \end{cases}$$

$$C'(l) = \begin{cases} c(1,1,l) & \text{if } l \neq 3 \\ 0 & \text{if } l = 3, \quad \mu = 0 \\ 1 & \text{if } l = 3, \quad \mu = 2 \text{ and } \nu(3) = 1 \\ 2 & \text{if } l = 3, \quad \mu = 2 \text{ and } \nu(3) = 2 \\ 0 & \text{if } l = 3, \quad \mu = 2 \text{ and } \nu(3) \geq 3 \\ 1 & \text{if } l = 3, \quad \mu = \infty \text{ and } \nu(3) = 1 \\ 2 & \text{if } l = 3, \quad \mu = \infty \text{ and } \nu(3) = 2 \\ 6 & \text{if } l = 3, \quad \mu = \infty \text{ and } \nu(3) \geq 3 \end{cases}$$

and

$$c(1,1,l) = \begin{cases} 2 & \mu(Q_l(\sqrt{-3}), L(l)) = 1 \text{ and } \quad \nu(l) = 1 \\ 2 & \mu(Q_l(\sqrt{-3}), L(l)) = \infty \\ 0 & \text{otherwise} \end{cases}$$

Here the product is over all distinct primes l dividing $\frac{N}{q}$ and $(\frac{*}{*})$ is the Kronecker symbol. In particular, $(\frac{-3}{3}) = (\frac{-4}{2}) = 0$ and $(\frac{-3}{2}) = -1$. Also, $\mu = \mu(L(3), Q_3(\sqrt{-3}))$.

PROOF. From the definition of the Brandt matrix, we see that $H(N') = tr(B(1:N'))$ (see Remark 2.25 [14]). Let us calculate $tr(B(1:N'))$. By Theorem 3.9, if M is an order of level N', then

$$tr(B(1:N')) = \sum_s \sum_f \frac{1}{2} b(s,f) \prod_{l|N} c(s,f,l) + \text{Mass}(M)\,.$$

Here, we need to explain $b(s,f)$ and $c(s,f,l)$ first. Let η be a canonical image of x in $Q[x]/(x^2+sx+1)$. Then for each f, there is uniquely determined order $\mathcal{O}_f$ containing $Z + Z\eta$ as a submodule of index f. Let $h(\mathcal{O}_f)(w(\mathcal{O}_f))$ denote the number of locally principal $\mathcal{O}_f$ ideals (resp. $\frac{1}{2|U(\mathcal{O}_f)|}$). Then $b(s,f) = \frac{h(\mathcal{O}_f)}{w(\mathcal{O}_f)}$ Also $c(s,f,l)$ is the number of $M_l^\times = R_{\nu(l)}^\times(L(l))$ (see Definition 2.1) equivalence classes of optimal embeddings of $l^m\alpha$ into $M_l = R_{\nu(l)}(L(l))$ where $Z + Z\alpha$ is the maximal order of $Q[x]/(x^2+sx+1)$ and $\mathcal{O}_f \otimes Z_l = Z_l + Z_l l^m \alpha$.

As $Q[x]/(x^2+sx+1)$ is a quadratic imaginary number field, $s^2-4<0$. Hence, there are three choices for s. Namely, $s=0$ or 1 and -1. However, since $Q[x]/(x^2+x+1) \simeq Q[x]/(x^2-x+1) \simeq Q(\sqrt{-3})$, it suffices to consider only the cases, $s=0$ and 1.

i) case $s=0$. (i.e. $s^2-4n=-4$).
Let $K=Q[x]/(x^2+1) \simeq Q(\sqrt{-1})$. Then $Z+Z\sqrt{-1}$ is the maximal order of K. So $f=1$. Let $\mathcal{O}=Z+Z\sqrt{-1}$ for convenience.

Now we need to find $b(0,1)$ of $\mathcal{O}$.

By [23; p267], the class number of $\mathcal{O}$ is 1 and the number of units in $\mathcal{O}$ is 4. That is, $h(\mathcal{O})=1$ and $w(\mathcal{O})=\frac{1}{2}|U(\mathcal{O})|=2$. Hence $b(0,1)=\frac{h(\mathcal{O})}{w(\mathcal{O})}=\frac{1}{2}$.

Next we need to calculate $c(s,f,l)$ for $l|N$.

First, if $l=q$, then $c(0,1,q)=(1-(\frac{-4}{q}))$ is given in Proposition 6 [4; p102].

Second, consider $l|\frac{N}{q}$. $\mathcal{O}_1 \otimes Z_l=(Z+Z\sqrt{-1})\otimes Z_l=Z_l+Z_l\sqrt{-1}$.

$\Delta(\sqrt{-1})=-4$ implies that $Z_l+Z_l\sqrt{-1} \simeq Z_l \oplus Z_l$ or $Z_l+Z_l\sqrt{-1}$ is the ring of integers in a field $Q_l(\sqrt{-1})$.

If $Z_l+Z_l\sqrt{-1} \simeq Z_l \oplus Z_l$, then since $L(l)$ is a field, by Theorem 3.10 in [6] $\mu(Q_l(\sqrt{-1}),L(l))=0$ or 1. By Theorem 5.30 and 5.31 in [6], $c(0,1,l)$, the number of $M_l^\times = R_{\nu(l)}^\times(L(l))$ equivalence classes of optimal embeddings of $\sqrt{-1}$ into $M_l=R_{\nu(l)}(L(l))$ is 2 if $L(l)$ is ramified and $\nu(l)=1$, i.e. $\mu(Q_l(\sqrt{-1}),L(l))=1$ and $\nu(l)=1$. Otherwise, by Theorem 5.19 and Table 5.28 in [6] $c(0,1,l)=0$. If, on the other hand, $Z_l+Z_l\sqrt{-1}$ is the ring of integers in a field $Q_l(\sqrt{-1})$, then since $2\nmid\frac{N}{q}$, $l\nmid\Delta(\sqrt{-1})=-4$. So $Q_l(\sqrt{-1})$is unramified. By Theorem 5.19 in [6], $c(0,1,l)=2$ if $L(l)$ is unramified, that is $\mu(Q_l(\sqrt{-1}),L(l))=\infty$. Otherwise, by Theorem 5.19 and Table 5.28 in [6] $c(0,1,l)=0$.

Hence

$$c(0,1,l)=\begin{cases} 2 & \text{if} \quad \mu(Q_l(\sqrt{-1}),L(l))=1 \quad \text{and } \nu(l)=1 \\ 2 & \text{if} \quad \mu(Q_l(\sqrt{-1}),L(l))=\infty \\ 0 & \text{otherwise} \end{cases}.$$

ii) case $s = 1$. (i.e. $s^2 - 4n = -3$).
Let $K = Q[x]/(x^2 + x + 1) = Q(\sqrt{-3})$. Then $Z + Z\sqrt{-3}$ is the maximal order of K. Hence, $f = 1$. Let $\mathcal{O} = Z + Z\sqrt{-3}$ for convenience.

The class number of $\mathcal{O}$ is 1 and the number of units in $\mathcal{O}$ is 6 (see [19; p267]). Hence $b(1,1) = \frac{h(\mathcal{O})}{w(\mathcal{O})} = \frac{1}{3}$ and we obtain $c(1,1,1)$ as in the theorem by the table 5.28 in [6].

Again, we need to calculate $c(s,f,l)$ for $l|N$.

First, if $l = q$, then $c(1,1,q) = (1 - (\frac{-3}{q}))$ was calculated by Eichler [2; p102].

Second, if $l|\frac{N}{q}$ and $l \neq 3$, then $c(1,1,l)$ is the number of $M_l^\times = R_{\nu(l)}^\times(L(l))$ equivalence classes of optimal embeddings of $\sqrt{-3}$ into $M_l = R_{\nu(l)}(L(l))$.

Since $\Delta(\sqrt{-3}) = -12$, $Q_l(\sqrt{-3})$ is either unramified or isomorphic to $Q_l \oplus Q_l$.

Analogous to the case i), by Theorem 5.19, 5.30, 5.31 and Table 5.28 in [6], $c(1,1,l)$ is calculated as in the theorem.

Finally, if $l|\frac{N}{q}$ and $l = 3$, since $\Delta(\sqrt{-3}) = -12 = -3 \cdot 4$, $Q_l(\sqrt{-3})$ is ramified. By table 5.28 and Theorem 5.19 in [6],

$$c(1,1,3) = \begin{cases} 0 & \text{if} \quad \mu = 0 \\ 1 & \text{if} \quad \mu = 2 \text{ and } \nu(3) = 1 \\ 2 & \text{if} \quad \mu = 2 \text{ and } \nu(3) = 2 \\ 0 & \text{if} \quad \mu = 2 \text{ and } \nu(3) \geq 3 \\ 1 & \text{if} \quad \mu = \infty \text{ and } \nu(3) = 1 \\ 2 & \text{if} \quad \mu = \infty \text{ and } \nu(3) = 2 \\ 6 & \text{if} \quad \mu = \infty \text{ and } \nu(3) \geq 3 \end{cases}$$

where $\mu = \mu(L(3), Q_3(\sqrt{-3}))$ (see Definition 3.3).

Combining i) and ii), we obtain that

$$\sum_{s}\frac{1}{2}\sum_{f} b(s,f)\prod_{l|N} c(s.f.l)$$
$$=\frac{1}{2}b(0,1)\prod_{l|N} c(0,1,l)$$
$$+\frac{1}{2}b(1,1)\prod_{l|N} c(1,1,l)+\frac{1}{2}b(-1,1)\prod_{l|N} c(-1,1,l)$$
$$=\frac{1}{4}(1-(\frac{-4}{q}))\prod_{l|\frac{N}{q}} C(l)+\frac{1}{3}(1-(\frac{-3}{q})\prod_{l|\frac{N}{q}} C'(l).$$

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