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NORMAL QUINTIC ENRIQUES SURFACES WITH MODULI NUMBER 6

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ABSTRACT. In this paper, we show one family of normal quintic surfaces in $\mathbf{P}^3$ which are birationally isomorphic to Enriques surfaces. We prove that the dimension of the moduli space of these Enriques surfaces is 6.

1. Introduction

1.1. An Enriques surface S is a non-singular surface S over a complex number field $\mathbb{C}$ satisfying one of the following equivalent conditions **[1]**, **[2]**, **[7]**, **[8]**:

(1) $2K_S \sim \leq_S$, but $K_S \not\sim \leq_S$, and $q(S) = 0$.
(2) $K_S \equiv 0$ and $b_2(S) = 10$.
(3) S is minimal with $\varkappa(S) = 0$ and $b_2(S) = 10$.
(4) S is minimal with $\varkappa(S) = 0$ and $p_g = 0$, $q = 0$.

Normal quintic Enriques surfaces are then normal quintic surfaces in $\mathbf{P}^3$ which are birationally isomorphic to Enriques surfaces.

We present a family of normal quintic surfaces in $\mathbf{P}^3$, say $\mathcal{F}$, which are birationally isomorphic to Enriques surfaces. These Enriques surfaces are characterized by a special type of divisors D. Our main concern is to show that the space of Enriques surfaces obtained from the family of normal quintic surfaces $\mathcal{F}$ is of dimension 6.

1.2. We now discuss on singularities, especially on minimally elliptic singularities. First let us give a definition of a geometric genus of a singular point.

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DEFINITION. The geometric genus $h(p)$ of V at p is the dimension of the complex vector space $H^1(M, \leq_M)$. This number is finite and independent of the choice of resolution of singularity $\pi : M \to V$. It may alternately be defined as the dimension of the stalk at the point p of the sheaf $R^1\pi_*(\leq_M)$ on V, which is concentrated at the point p.

If $h(p) = 0$, then p is called a rational singularity. A rational singularity embeds in codimension 1 if and only if it is a double point. And among all surface singularities rational double points are the simplest ones. They are classified into the following five types with well-known dual graphs :

$$\begin{array}{rl} A_n(n \geq 1) & z^2 + x^2 + y^{n+1} = 0 \\ D_n(n \geq 4) & z^2 + y(x^2 + y^{n-2}) = 0 \\ E_6 & z^2 + x^3 + y^4 = 0 \\ E_7 & z^2 + x(x^2 + y^3) = 0 \\ E_8 & z^2 + x^3 + y^5 = 0 \end{array}$$

DEFINITION. A cycle $D > 0$ on X is rational if $\chi(\mathcal{O}_X(D)) = 1$, elliptic if $\chi(\mathcal{O}_X(D)) = 0$, and minimally elliptic if $\chi(\mathcal{O}_X(D)) = 0$ and $\chi(\mathcal{O}_X(C)) > 0$ for all cycles C such that $0 < C < D$. Let Z_A be the fundamental cycle of an isolated singular point $p \in X$, then p is called rational (weakly elliptic, minimally elliptic) point if Z_A is rational (elliptic, minimally elliptic).

In **[10]**, H. Laufer shows that a point p is minimally elliptic if and only if $h(p) = 1$ and its local ring $\leq_{V,p}$ is Gorenstein. Since a hypersurface singularity is Gorenstein, a singular point p of a hypersurface in $\mathbf{P}^3$ is minimally elliptic if and only if $h(p) = 1$. Let Z be a fundamental cycle of a minimally elliptic point p. Then if $Z \cdot Z = -1$ or -2, then p is a double point, and if $-3 \leq Z \cdot Z \leq -1$, then the point p is a hypersurface singularity (Theorem 3.13, **[10]**). If p is a minimally elliptic singularity which is not a double point, then p is an absolutely isolated point, that is, a singularity which can be resolved by blowing up points alone (Theorem 3.15, **[10]**). He also gives a list of defining equations and dual graphs of all minimally elliptic double and triple points. In his list, if the exceptional set A is a non-singular elliptic curve, then

(1) if $Z \cdot Z = -1$, then the equation of p is $T_{2,3,6} \,:\, z^2 + x^3 + y^6 = 0$.
(2) if $Z \cdot Z = -2$, then the equation of p is $T_{2,4,4} \,:\, z^2 + x^4 + x^2y^2 + y^4 = 0$.
(3) if $Z \cdot Z = -3$, then the equation of p is $T_{3,3,3} \,:\, x^3 + y^3 + z^3 = 0$.

From one of the fifteen equivalent characterizations by Alan Durfee, rational double points are also absolutely isolated singular points (**[5]**). On the other hand minimally elliptic double points may be characterized as those points such that they could be resolved by blowing up double points alone except exactly one step where the blow-up is along a double curve. This can be checked by resolving the equations of minimally elliptic double points (provided by H. Laufer, **[10]**).

According to the classical definition, a point p of a surface X is called "tacnode" if it is a double point and X has an infinitesimal double line **L** in the first neighborhood of p (page 426, **[11]**). Following this definition, all minimally elliptic double points are tacnodes. In this paper, we will consider only those tacnodes which are minimally elliptic double points.

DEFINITION. Tacnodes are minimally elliptic double points with $Z \cdot Z = -2$.

Throughout this paper, we will say simply tacnodes of type Γ ignoring self-intersection numbers of irreducible components of Γ. Particularly, tacnodes of type I_0, which appear generically, are simple elliptic singularities,

$$T_{2,4,4} \,:\, z^2 + x^4 + y^4 + ax^2y^2 \,,\ a^4 \neq 4$$

whose exceptional sets are non-singular elliptic curves.

Most of tacnodes we will treat in this paper are tacnodes of type I_n, $0 \leq n \leq 9$, which are also cusp singularities. In general, the equations of tacnodes are given as follows :

$$z^2 + f(x, y) = 0\,,$$

where $f(x, y)$ are polynomials of degree 4 or 5. We define a *tacnodal plane* to be the plane given by the equation $z = 0$ in the above equation.

2. Normal quintic Enriques surfaces

2.1. We first show Ezio Stagnaro's claim employing a modern language, which states that a family of special normal quintic surfaces in $\mathbf{P}^3$, say $\mathcal{F}$, are birationally isomorphic to Enriques surfaces. Then we investigate the condition on Enriques surfaces which are birationally isomorphic to the above special normal quintic surfaces of $\mathcal{F}$.

However, since our emphasis in this paper is showing that Enriques surfaces obtained from normal quintic surfaces of $\mathcal{F}$ are of moduli number 6, most of proofs in this section will be brief, and the detailed proofs will appear at the coming paper on normal quintic Enriques surfaces treating mainly the second family of normal quintic surfaces (**cf.** [**9**]).

THEOREM 2.1. (Ezio Stagnaro [**12**]) *Let F_5 be a normal quintic surface in $\mathbf{P}^3$ with the following property, say $\mathcal{P}$:*

F_5 has four tacnodal points at the vertices A_1, A_2, A_3, A_4 of a tetrahedron T such that there exist two planes α_1, α_2, where α_1 is the tacnodal plane to F_5 at A_1, A_2 and α_2 the tacnodal plane to F_5 at A_3, A_4.

If S is a minimal non-singular model of F_5, then S is an Enriques surface.

PROOF. Let $\tilde{S}$ be a minimal desingularization of the surface F_5. Then we show that $p_g(\tilde{S}) = 0$, $q(\tilde{S}) = 0$ and $P_2(\tilde{S}) = 1$. It is easy to show that $\varkappa(\tilde{S}) = 0$. Then from the classification of surfaces with $\varkappa = 0$, the minimal model S of $\tilde{S}$ is an Enriques surface.

To show that $P_2(\tilde{S}) = 1$, we use the following fact which is essential in understanding the reason why we have to impose a special condition on two tacnodal planes α_1 and α_2 : Let $\tilde{e} \subset \tilde{V}$ be the exceptional set of the minimal desingularization $\sigma : \tilde{V} \longrightarrow V$, where V is an affine neighborhood of one of two tacnodes. Then the tacnodal plane H_0 is the unique hypersurface section of V whose the total transform, $\sigma^*(H_0) = \tilde{H}_0 + 2\tilde{e}$; and for all other hypersurface sections H containing the tacnode as a regular point, $\sigma^*(H) = \tilde{H} + \tilde{e}$.

From this remark, it is easy to see that $\alpha_1 + \alpha_2$ is the only effective divisor of $|2K_{\tilde{S}}|$, hence $P_2(\tilde{S}) = 1$. □

COROLLARY 2.2. *Let X be a minimal non-singular model of a normal quintic surface F_5 which has four tacnodes in general position and does not satisfy the property $\mathcal{P}$. Then X is a rational surface.*

PROOF. Similarly to the proof of Theorem 2.1, it is easy to find that $p_g(X) = 0$, $q(X) = 0$ and $P_2(X) = 0$, which implies $K_X \cdot K_X \leq -1$. Hence $\varkappa(X) = -\infty$ and X is a rational surface. □

2.2. We now fix four points of the tetrahedron T, say $A_1 = (1, 0, 0, 0)$, $A_2 = (0, 0, 1, 0)$, $A_3 = (0, 1, 0, 0)$, $A_4 = (0, 0, 0, 1)$, and two tacnodal planes to F_5,

$$\alpha_1 : x_1 + x_3 = 0 \text{ and } \alpha_2 : x_2 + x_4 = 0$$

at A_3, A_4 and A_1, A_2 respectively.

PROPOSITION 2.3. *F_5 contains three lines L_1, L'_1 and L_2 ; the lines $L_1 = \overline{A_1A_2}$ and $L'_1 = \overline{A_3A_4}$ are lines joining two vertices of the tetrahedron T and L_2 is the intersection of two tacnodal planes α_1 and α_2. Furthermore, the normal quintic surface F_5 has the following equation :*

$$\begin{aligned} F_5 \ : \ & (x_2^3 + x_4^3)(x_1 + x_3)^2 \\ & +(x_1^3 + x_3^3)(x_2 + x_4)^2 \\ & +(a_1x_1x_2x_3 + a_2x_1x_2x_4 + a_3x_1x_3x_4 + a_4x_2x_3x_4)(x_1 + x_3)(x_2 + x_4) \\ & +a_5x_2^2x_4^2(x_1 + x_3) + a_6x_1^2x_3^2(x_2 + x_4) = 0\, ; \ a_5 \neq 0\, , \ a_6 \neq 0. \end{aligned}$$

PROOF. The line L_1 joining A_1 and A_2 meets F_5 with multiplicity 4 at A_1 and A_2 because L_1 belongs to α_2, and α_2 cuts out F_5 a divisor D_1, which is a hyperplane section of F_5, with multiplicity 4 at A_1 and A_2. Hence L_1 has to be in F_5 and the line L_1 has multiplicity 4 in the divisor D_1, that is, $D_1 = 4L_1 + \ell_1$, ℓ_1 a line. Similarly, we may say that L'_1 is in F_5 and α_1 cuts out a divisor D'_1 with $D'_1 = 4L'_1 + \ell'_1$, ℓ'_1 a line. The line L_2, which is the intersection of α_1 and α_2, meets L_1 and L'_1 with multiplicity 4 because L_2 belongs to α_1, α_2, and α_1, α_2 cuts out F_5 hyperplane sections which contain L_1 and L'_1 with multiplicity 4. Hence L_2 is also in F_5 and is a component of both D_1 and D'_1. Therefore $\ell_1 = \ell'_1 = L_2$, and $D_1 = 4L_1 + L_2, D'_1 = 4L'_1 + L_2$.

Now let us find the equation of F_5. First write

$$F_5 : (x_1 + x_3)S + (x_2 + x_4)T + R_5,$$

where S and T are polynomials of degree 4, and R_5 a polynomial of degree 5 which does not have any term divisible by (x_1+x_3) or (x_2+x_4). We follow the following rule to choose S and T : Let $(x_1 + x_3)^m(x_2 + x_4)^n K$ be a degree 5 term of F_5 for some K. Then $(x_1 + x_3)^{m-1}(x_2 + x_4)^n K$ is in S if $m \geq n$, and $(x_1 + x_3)^m(x_2 + x_4)^{n-1}K$ is in T if $m \leq n$.

Let $S = S_4 + (x_1 + x_3)S_3 + (x_2 + x_4)U_3$ and $T = T_4 + (x_2 + x_4)T_3 + (x_1 + x_3)V_3$, where polynomials of degree 4 S_4, T_4 do not have any term divisible by $(x_1 + x_3)$ or $(x_2 + x_4)$. U_3 is a polynomial of degree 3 which does not have any term divisible by $(x_1 + x_3)$, and similarly V_3 is a polynomial of degree 3 which does not have any term divisible by $(x_2 + x_4)$. Let $W_3 = U_3 + V_3$. Then we get

$$\begin{aligned} F_5 : (x_1 + x_3)S_4 &+ (x_2 + x_4)T_4 + (x_1 + x_3)^2 S_3 + (x_2 + x_4)^2 T_3 \\ &+ (x_1 + x_3)(x_2 + x_4)W_3 + R_5. \end{aligned}$$

F_5 satisfies the following conditions :

(1) F_5 has multiplicity 2 at points A_1, A_2, A_3 and A_4

(2) $(F_5 = 0$ and $(x_1+x_3) = 0)$, which is $(x_2+x_4)T_4+(x_2+x_4)^2T_3+R_5 = 0$, has multiplicity 4 at A_3 and A_4. Similarly, $(F_5 = 0$ and $(x_2+x_4) = 0)$, which is $(x_1+x_3)S_4+(x_1+x_3)^2S_3+R_5 = 0$, also has multiplicity 4 at A_1 and A_2.

(3) The leading terms of $F_5|_{x_1=1}$ and $F_5|_{x_3=1}$ are $(x_2 + x_4)^2$. Similarly, the leading terms of $F_5|_{x_2=1}$ and $F_5|_{x_4=1}$ are $(x_1 + x_3)^2$.

From (2), S_3 should be divisible by $(x_2 + x_4)$ because S_3 is a polynomial of degree 3. Similarly, T_3 is divisible by $(x_1 + x_3)$. Then condition (2) is equivalent to the following :

(2′) $(F_5 = 0$ and $(x_1+x_3) = 0)$, which is $(x_2+x_4)T_4+R_5 = 0$, has the multiplicity 4 at A_3 and A_4. Similarly, $(F_5 = 0$ and $(x_2 + x_4) = 0)$, which is $(x_1 + x_3)S_4 + R_5 = 0$, also has the multiplicity 4 at A_1 and A_2.

From (1), S_4 can not have monomials x_2^4, x_4^4 and also can not have x_1^4, x_3^4 because otherwise A_1 and A_2 do not belong to F_5. Similarly, T_4 can not have monomials x_1^4, x_2^4, x_3^4 and x_4^4.

S_4 can not have monomials $x_1^3\ell, x_3^3\ell$ for a linear form ℓ because of condition (1). S_4 also can not have monomials $x_2^3\ell, x_4^3\ell$ for a linear form ℓ because of condition (3). From (2′), S_4 can not have monomials containing x_1 or x_3. Hence S_4 has only one monomial $x_2^2x_4^2$. Similarly, T_4 has only one monomial $x_1^2x_3^2$.

S_3 can not have monomial $x_2x_4^2, x_2^2x_4$ and any monomials containing x_1 or x_3 because of condition (3). Hence the only possible monomials for S_3 are x_2^3 and x_4^3. Then $S_3 = x_2^3 + x_4^3$ up to constant. Similarly, $T_3 = x_1^3 + x_3^3$ up to constant.

From (3), the only monomials of W_3 are $x_1x_2x_3, x_1x_2x_4, x_1x_3x_4$ and $x_2x_3x_4$.

Obviously R_5 can not have monomials $x_1^5, x_2^5, x_3^5, x_4^5$. R_5 can not have monomials $x_1^4x_2, x_1^4x_3, \ldots, x_4^4x_2, x_4^4x_3$ because of condition (1). From condition (3) R_5 can not have $x_1^3x_2x_3, x_1^3x_2x_4, \ldots, x_4^3x_3x_1, x_4^3x_3x_2$. Hence from condition (2) R_5 must have, if any, monomials of the following types, $x_1p_4, x_2q_4, x_3s_4, x_4t_4$, where p_4, q_4, s_4, t_4 are polynomials of degree 4. However, if R_5 has the monomial, for example, $x_1(x_2^2x_3^2)$, then $(F_5 = 0 \,\text{and}\, (x_1 + x_3) = 0)$ would not have multiplicity 4 at A_2. Hence R_5 must be zero.

Therefore we get the following equation :

$$
\begin{aligned}
F_5 \ :\ & a(x_2^3 + x_4^3)(x_1 + x_3)^2 \\
& + b(x_1^3 + x_3^3)(x_2 + x_4)^2 \\
& + \left(c_1x_1x_2x_3 + c_2x_1x_2x_4 + c_3x_1x_3x_4 + c_4x_2x_3x_4\right)(x_1 + x_3)(x_2 + x_4) \\
& + c_5x_2^2x_4^2(x_1 + x_3) + c_6x_1^2x_3^2(x_2 + x_4) = 0, \\
& \text{where} \quad a \neq 0, b \neq 0, c_5 \neq 0 \,\text{and}\, c_6 \neq 0.
\end{aligned}
$$

Multiply the above equation by a^2b^2 as follows :

$$
\begin{aligned}
& a^3b^2(x_2^3 + x_4^3)(x_1 + x_3)^2 + a^2b^3(x_1^3 + x_3^3)(x_2 + x_4)^2 \\
& + a^2b^2\left(c_1x_1x_2x_3 + c_2x_1x_2x_4 + c_3x_1x_3x_4 + c_4x_2x_3x_4\right) \\
& (x_1 + x_3)(x_2 + x_4)
\end{aligned}
$$

$$
\begin{aligned}
&+ a^2b^2c_5x_2^2x_4^2(x_1+x_3) + a^2b^2c_6x_1^2x_3^2(x_2+x_4)\\
=&\big((ax_2)^3+(ax_4)^3\big)(bx_1+bx_3)^2\\
&+\big((bx_1)^3+(bx_3)^3\big)(ax_2+ax_4)^2\\
&+ab\big(c_1x_1x_2x_3+c_2x_1x_2x_4+c_3x_1x_3x_4+c_4x_2x_3x_4\big)\\
&(bx_1+bx_3)(ax_2+ax_4)\\
&+a^2bc_5x_2^2x_4^2(bx_1+bx_3)+ab^2c_6x_1^2x_3^2(ax_2+ax_4)=0
\end{aligned}
$$

We apply the torus group action to get the linear equations $(x_1 + x_3), (x_2 + x_4)$ from $(bx_1 + bx_3), (ax_2 + ax_4)$ respectively. Then after adjusting coefficients, we get the desired equation of F_5. □

2.3. Let $\sigma : \tilde{S} \longrightarrow F_5$ be the minimal desingularization of F_5 and $\rho : \tilde{S} \longrightarrow S$ the blow-down of $\tilde{S}$ to the minimal model S, which is an Enriques surface. In this section we classify the Enriques surface S which is birationally isomorphic to a normal quintic surface F_5 in $\mathbf{P}^3$ with the property $\mathcal{P}$ of Theorem 2.1. For this purpose, we seek to find the divisor D on S which corresponds to a hyperplane section of F_5 by the birational isomorphism $f = \sigma \circ \rho^{-1} : S \longrightarrow F_5$ and plan to claim that every Enriques surface S with the divisor D and an additional property, which will be stated in Theorem 2.5, is birationally isomorphic to a normal quintic surface F_5 in $\mathbf{P}^3$ with the property $\mathcal{P}$.

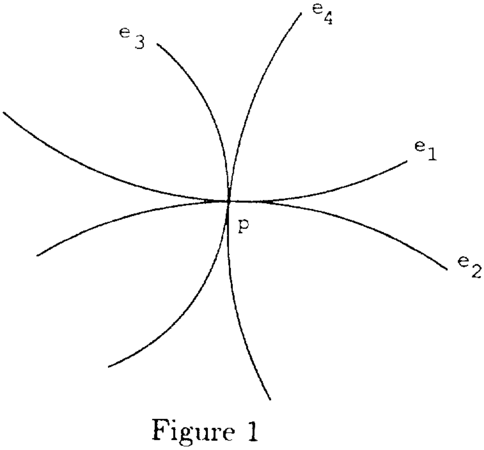

Figure 1

We note that the normal quintic surface F_5 contains three lines L_1, L_1', L_2. Let l_1, l_1' and l_2 be the proper transforms of lines L_1, L_1' and L_2 by the map $\sigma : \tilde{S} \longrightarrow F_5$. It is easy to see that l_1 and l_1' are exceptional curves of the first kind on $\tilde{S}$. If we blow down exceptional curves of the first kind l_1 and l_1' to smooth points, then l_2 becomes an exceptional curve of the first kind. Thus we could blow down a rational curve l_2 to a smooth point too. Let S be the surface obtained from $\tilde{S}$ after blowing down l_1, l_1' and l_2. Then $K_S \cdot K_S = 0$ since $K_{\tilde{S}} \cdot K_{\tilde{S}} = -3$. Hence the surface S is the minimal surface. Let $\tilde{H} = \sigma^*(H)$ be the proper transform of H, a hyperplane section of F_5 by the minimal desingularization map $\sigma : \tilde{S} \longrightarrow F_5$, and D the divisor on S which corresponds to $\tilde{H}$ by the composition map of the above three blowing downs.

After blowing down exceptional curves l_1, l_1' first and then l_2, we see that the divisor $\tilde{H}$ of $\tilde{S}$ corresponds to a divisor D with the configuration in Figure 1, where e_1, e_2, e_3, e_4 are isolated elliptic curves (or indecomposable divisors of canonical type) on S which are the images of $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$ and $\tilde{e}_4$ by the blowing-down map $\rho : \tilde{S} \longrightarrow S$ (We note that $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4$ are the exceptional sets on $\tilde{S}$ for the minimal desingularization map $\sigma : \tilde{S} \longrightarrow F_5$).

By summarizing what we have observed, we get the following proposition.

PROPOSITION 2.4. *If S is the Enriques surface obtained from the normal quintic surface F_5 satisfying the property $\mathcal{P}$ of Theorem 2.1, then S has a divisor $D = e_1 + e_2 + e_3 + e_4$ with the configuration in the Figure 1.*

Conversely we show that a generic Enriques surface S with a divisor $D = e_1 + e_2 + e_3 + e_4$ with the configuration in Figure 1 is birationally isomorphic to a normal quintic surface F_5 in $\mathbf{P}^3$ satisfying the property $\mathcal{P}$.

THEOREM 2.5. *Let S be an Enriques surface with a divisor $D = e_1 + e_2 + e_3 + e_4$ with the configuration in Figure 1, that is, e_1, e_2, e_3, e_4 are isolated elliptic curves and $e_1 \cdot e_3 = e_1 \cdot e_4 = e_2 \cdot e_3 = e_2 \cdot e_4 = 1$, and $e_1 \cdot e_2 = e_3 \cdot e_4 = 2$ with the following additional geometric property $\mathcal{GP}$:*

"e_1, e_2 and e_3, e_4 meet tangentially at the same point $p \in S$."

Then the following statements are true :

(1) *If the adjoints e_1', e_2', e_3' and e_4' do not have a common point, then S is birationally isomorphic to a normal quintic surface F_5 in $\mathbf{P}^3$ satisfying the property $\mathcal{P}$ of Theorem 2.1, where four tacnodes are of type I_n $(0 \le n \le 9)$, which are cusp singularities, with possibly finitely many isolated rational double points.*

(2) *If the adjoints e_1', e_2', e_3' and e_4' have a common point, then S is birationally two to one onto a quadric surface Q in $\mathbf{P}^3$.*

REMARK. Notice that e_1', e_2', e_3' and e_4' can not have more than one common point, if any, because $e_1' \cdot e_3' = e_1' \cdot e_4' = e_2' \cdot e_3' = e_2' \cdot e_4' = 1$.

REMARK. It is well known that every Enriques surface has a divisor $D = e_1 + e_2 + e_3 + e_4$ satisfying all conditions of Theorem 2.5 without the geometric property $\mathcal{GP}$. And for such a generic Enriques surface with the property $\mathcal{GP}$, e_1', e_2', e_3', e_4' do not have a common point.

PROOF. Let $\tilde{D} = \varphi^*(K_S) + 2\tilde{L}_1 + 2\tilde{L}_1' + \tilde{L}_2 + \tilde{e}_1 + \tilde{e}_2 + \tilde{e}_3 + \tilde{e}_4$, a divisor on $\tilde{S}$. We then show that the complete linear system $|\tilde{D}|$ determined by $\tilde{D}$ corresponds to a sublinear system $\mathcal{L}$ of a complete linear system on S, which induces the projection $\pi : S \to F_5$, a map from the Enriques surface S to a normal quintic surface F_5 in $\mathbf{P}^3$ with the property $\mathcal{P}$. The detailed proof is referred to [9].

3. The linear independence of four tacnodes

3.1. Let $f : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ be a germ and $M(f) = \mathbb{C}[x, y, z]/J_f$, where J_f is the jacobian ideal of f, that is, the ideal generated by the partial derivatives of f : f_x, f_y and f_z.

DEFINITION. The number $\mu(f) = \dim M(f)$ is called the Milnor number of f and $M(f)$ the Milnor algebra of f.

PROPOSITION 3.1. *Let $f : \mathbb{C}^n \to \mathbb{C}$ be a weighted homogeneous polynomial of degree d with respect to the weights $w = (w_1, \ldots, w_n)$. Then*

$$\mu(f) = \frac{(d - w_1) \ldots (d - w_n)}{w_1 \ldots w_n}$$

PROOF. For the proof of this proposition, we refer to [**5**]. □

Lastly we present the criterion for a function to have a tacnode of type I_0. First let $\varphi = z^2 + x^4 + ax^2y^2 + y^4$, $a^2 \neq 4$. Let X be a hypersurface in $\mathbf{P}^3$ given by a polynomial f, where the equation of f in an affine neighborhood of a fixed point p is

$$f = \sum_{0 \leq i+j+k \leq n} a_{ijk} x^i y^j z^k ,$$

where the point p corresponds to the origin.

Then we have the following criterion to have a tacnodal singularity of type I_0, that is, simple elliptic singularity $T_{2,4,4}$ at the point p.

LEMMA 3.2. *The necessary and sufficient condition for X to have a tacnodal singularity of type I_0 at p is that the coefficient of z, $a_{001} = 0$ and $\bar{f} = 0$, where $\bar{f} \in M(\varphi)$ is the representative of f in the Milnor algebra of φ.*

PROOF. Suppose that a polynomial function f given by the above equation satisfies the conditions, $a_{001} = 0$ and $\bar{f} = 0$. By analytic coordinate changes, we can eliminate all terms containing z-variable. Then f becomes the function φ added with terms of degree higher than five. Hence f has a tacnode of type I_0 at p. □

3.2. Let $\varphi = z^2 + x^4 + \lambda x^2y^2 + y^4$, $\lambda^2 \neq 4$. Then from Proposition 3.1, $\mu(\varphi) = 9$. Actually, it is easy to find generators of $M(\varphi)$ as follows :

$$\bar{1}, \bar{x}, \bar{y}, \bar{x}^2, \bar{x}\bar{y}, \bar{y}^2, \bar{x}^3, \bar{y}^3, \bar{x}^2\bar{y}^2, \tag{3–1}$$

where the bar over a letter denotes its representative in the Milnor algebra $M(\varphi)$.

NOTATION. For a given polynomial $f \in \mathbb{C}[x_1, x_2, x_3, x_4]$, let "$f|_{x_i=1}$" be the function obtained from f after taking $x_i = 1$, which may be considered as a polynomial function in the affine neighborhood $x_i = 1$.

PROPOSITION 3.3. *In the space of all quintic surfaces of* $\mathbf{P}^3$, *tacnodal singularities of type* I_0, *i.e. simple elliptic singularities* $T_{2,4,4}$ *at four points* P_1, P_2, P_3, P_4 *of* $\mathbf{P}^3$, *which are in general position, give 40 linearly independent conditions.*

PROOF. In this section, all tacnodes are assumed to be tacnodes of type I_0 or simple elliptic singularities

$$T_{2,4,4} \ : \ z^2 + x^4 + ax^2y^2 + y^4 \quad a^2 \neq 4.$$

We fix four tacnodal points to be the vertices of the coordinate tetrahedron T, that is, $P_1 = (1,0,0,0), P_2 = (0,1,0,0), P_3 = (0,0,1,0), P_4 = (0,0,0,1)$. Let us assume that we have chosen local affine coordinates x, y, z at each points P_1, P_2, P_3 and P_4. Then there corresponds a function φ_i at each point P_i which has the same equation as the above φ with the chosen local coordinates. Let $\mathbb{C}[x_1, x_2, x_3, x_4]|_{P_i}$ be the polynomial algebra in an affine neighborhood of P_i by taking $x_i = 1$. Then there is a projection mapping

$$\begin{aligned}\mathbf{\Phi} \ &: \mathbb{C}[x_1, x_2, x_3, x_4] \\ &\longrightarrow \left(\mathbb{C}^4, \mathbb{C}[x_1, x_2, x_3, x_4]|_{P_1}/J, \ldots, \mathbb{C}[x_1, x_2, x_3, x_4]|_{P_4}/J\right),\end{aligned}$$

where $\mathbb{C}[x_1, x_2, x_3, x_4]|_{P_i}/J$ is the Milnor algebra of the function φ_i defined in a neighborhood of P_i. The projection $\mathbf{\Phi}$ is defined as follows : for a given polynomial $f \in \mathbb{C}[x_1, x_2, x_3, x_4]$,

$$\mathbf{\Phi}(f) = (f|_{z,1}, \ldots, f|_{z,4}, \bar{f}|_{x_1=1}, \ldots, \bar{f}|_{x_4=1}),$$

where

$$\bar{f}|_{x_i=1} \in \mathbb{C}[x_1, x_2, x_3, x_4]|_{P_i}/J$$

is a representative of $f|_{x_i=1}$ in the corresponding Milnor algebra, and "$f|_{z,i}$" is the coefficient of z variable of the function $f|_{x_i=1}$ where z is the degree two variable in the equation of φ defined in an affine neighborhood of P_i.

We have the expression of the polynomial algebra $\mathbb{C}[x_1, x_2, x_3, x_4]$ as a summation of graded homogeneous polynomial algebras :

$$\mathbb{C}[x_1, x_2, x_3, x_4] = \sum_{k=1}^{\infty} H^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(k))$$

Then we get the projection mapping $\mathbf{\Phi}_5$ which is the restriction of the projection $\mathbf{\Phi}$ to the subspace $H^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(5))$, the space of homogeneous polynomials of degree 5 :

$$\begin{aligned}\mathbf{\Phi}_5 :& H^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(5)) \\ &\to \left(\mathbb{C}^4, \mathbb{C}[x_1, x_2, x_3, x_4]|_{P_1}/J, \ldots, \mathbb{C}[x_1, x_2, x_3, x_4]|_{P_4}/J\right)\end{aligned}$$

Since $\mathbb{C}[x_1, x_2, x_3, x_4]|_{P_i}/J$ is a vector space of dimension 9 with generators given in (3–1), the projection $\mathbf{\Phi}_5$ is a linear mapping from a 56-dimensional vector space to a 40-dimensional vector space.

To prove Proposition 3.3., it is enough to show that the projection $\mathbf{\Phi}_5$ is surjective. Then from Lemma 3.2, a quintic surface F_5 with its defining equation f has tacnodal singularities at P_1, P_2, P_3, P_4 if and only if $\mathbf{\Phi}_5(f) = 0$. To show the surjectivity of $\mathbf{\Phi}_5$, we write a general homogeneous polynomial f of degree 5 as follows :

$$\begin{aligned}f :\ & a_1x_1^5 + a_2x_2^5 + a_3x_3^5 + a_4x_4^5 \\
&+a_5x_1^4x_2 + a_6x_1^4x_3 + a_7x_1^4x_4 + a_8x_1x_2^4 + a_9x_2^4x_3 + a_{10}x_2^4x_4 \\
&+a_{11}x_1x_3^4 + a_{12}x_2x_3^4 + a_{13}x_3^4x_4 + a_{14}x_1x_4^4 + a_{15}x_2x_4^4 + a_{16}x_3x_4^4 \\
&+a_{17}x_1^3x_2x_3 + a_{18}x_1^3x_2x_4 + a_{19}x_1^3x_3x_4 + a_{20}x_1^3x_2^2 + a_{21}x_1^3x_3^2 + a_{22}x_1^3x_4^2 \\
&+a_{23}x_1x_2^3x_3 + a_{24}x_1x_2^3x_4 + a_{25}x_2^3x_3x_4 + a_{26}x_1^2x_2^3 + a_{27}x_2^3x_3^2 + a_{28}x_2^3x_4^2 \\
&+a_{29}x_1x_2x_3^3 + a_{30}x_1x_3^3x_4 + a_{31}x_2x_3^3x_4 + a_{32}x_1^2x_3^3 + a_{33}x_2^2x_3^3 + a_{34}x_3^3x_4^2 \\
&+a_{35}x_1x_2x_4^3 + a_{36}x_1x_3x_4^3 + a_{37}x_2x_3x_4^3 + a_{38}x_1^2x_4^3 + a_{39}x_2^2x_4^3 + a_{40}x_3^2x_4^3 \\
&+a_{41}x_1^2x_2^2x_3 + a_{42}x_1^2x_2x_3^2 + a_{43}x_1^2x_2^2x_4 \\
&+a_{44}x_1^2x_2x_4^2 + a_{45}x_1^2x_3^2x_4 + a_{46}x_1^2x_3x_4^2 \\
&+a_{47}x_2^2x_3^2x_4 + a_{48}x_2^2x_3x_4^2 + a_{49}x_1x_2^2x_3^2\end{aligned}$$

$$+a_{50}x_1x_2^2x_4^2 + a_{51}x_1x_3^2x_4^2 + a_{52}x_2x_3^2x_4^2$$
$$+a_{53}x_1^2x_2x_3x_4 + a_{54}x_1x_2^2x_3x_4 + a_{55}x_1x_2x_3^2x_4 + a_{56}x_1x_2x_3x_4^2$$

Now we claim that every 40 coordinates of the image $\mathbf{\Phi}_5(f)$ has its unique coefficient of f, that is, we can find a_{i_j} from each j-th coordinate of $\mathbf{\Phi}_5(f)$ so that if $j \neq k$, then $i_j \neq i_k$. Then for a given element

$$\mathbf{v} \in \left(\mathbb{C}^4, \mathbb{C}[x_1, x_2, x_3, x_4]|_{P_1}/J, \ldots, \mathbb{C}[x_1, x_2, x_3, x_4]|_{P_4}/J\right),$$

we can find a homogeneous polynomial $f \in H^0(\mathbf{P}^3, O_{\mathbf{P}^3}(5))$ so that $\mathbf{\Phi}_5(f) = \mathbf{v}$ after adjusting chosen free coefficients. Hence the projection $\mathbf{\Phi}_5$ is surjective. Notice that the mapping $\mathbf{\Phi}_5$ depends on the choice of the local coordinates at each points P_i $(i = 1, \ldots, 4)$. For instance, if we take the local coordinates as follows :

$$x = x_2, y = x_3, z = x_4 \text{ at } P_1$$
$$x = x_3, y = x_4, z = x_1 \text{ at } P_2$$
$$x = x_4, y = x_1, z = x_2 \text{ at } P_3$$
$$x = x_1, y = x_2, z = x_3 \text{ at } P_4$$

then

$$f_{z,1} = a_7 \quad f_{z,2} = a_8 \quad f_{z,3} = a_{12} \quad f_{z,4} = a_{16}$$

$$\begin{aligned}
\bar{f}|_{x_1=1} =& (a_1, a_5\bar{x}, a_6 y, a_{20}\bar{x}^2, a_{17}xy, a_{21}\bar{y}^2, (a_{26} - 2/\lambda\ a_{42})\bar{x}^3,\\
& (a_{32} - 2/\lambda\ \mathbf{a}_{41})y^3, (-\lambda/2\ a_{11} + \mathbf{a}_{49} - \lambda/2\ a_8)\bar{x}^2\bar{y}^2)\\
\bar{f}|_{x_2=1} =& (a_2, a_9\bar{x}, a_{10}y, a_{27}\bar{x}^2, a_{25}x\bar{y}, a_{28}\bar{y}^2, (a_{33} - 2/\lambda\ a_{48})\bar{x}^3,\\
& (a_{39} - 2/\lambda\ \mathbf{a}_{47})\bar{y}^3, (-\lambda/2\ a_{15} + \mathbf{a}_{52} - \lambda/2\ a_{12})\bar{x}^2\bar{y}^2)\\
\bar{f}|_{x_3=1} =& (a_3, a_{13}x, a_{11}y, a_{34}\bar{x}^2, a_{30}x\bar{y}, a_{32}\bar{y}^2, (a_{40} - 2/\lambda\ a_{45})x^3,\\
& (a_{21} - 2/\lambda\ \mathbf{a}_{51})\bar{y}^3, (-\lambda/2\ a_{16} + \mathbf{a}_{46} - \lambda/2\ a_6)\bar{x}^2\bar{y}^2)\\
\bar{f}|_{x_4=1} =& (a_4, a_{14}x, a_{15}y, a_{38}\bar{x}^2, a_{35}xy, a_{39}\bar{y}^2, (a_{22} - 2/\lambda\ a_{50})\bar{x}^3,\\
& (a_{28} - 2/\lambda\ \mathbf{a}_{44})\bar{y}^3, (-\lambda/2\ a_{10} + \mathbf{a}_{43} - \lambda/2\ a_7)\bar{x}^2\bar{y}^2)
\end{aligned}$$

We choose coefficients a_i with bold letters only if a coordinate has such a coefficient in it. Then we can find 40 independent coefficients a_i from the above equations.

It is easy to see that from the invertible linear transformation which changes the coordinates of $\mathbf{P}^3$, there corresponds an invertible linear transformation acting on the coefficients of the polynomial f, that is, on a_i's. Hence, in a neighborhood of the identity in $GL(4)$, the mapping $\mathbf{\Phi}_5$ is still surjective. Hence four tacnodal points at P_1, P_2, P_3, P_4 give 40 linearly independent conditions on quintic surfaces in $\mathbf{P}^3$. □

REMARK. There is a similar but more general result on rational double points of hypersurfaces in $\mathbf{P}^3$ by Daniel Burns and Jonathan Wahl **[3]**. It is likely that four tacnodes of type I_0 are a maximum number on quintic surfaces in $\mathbf{P}^3$ which give linearly independent conditions, and we do not expect the same result for other types of tacnodes since tacnodes of type I_0 are generic with $\mu = 9$.

THEOREM 3.4. *Let $\mathcal{Q}$ be the moduli space of normal quintic surfaces in $\mathbf{P}^3$ which satisfy the property $\mathcal{P}$ of Theorem 2.1. Then the dimension of the moduli space $\mathcal{Q}$ is 6.*

PROOF. First we count normal quintic surfaces F_5 in $\mathbf{P}^3$ with tacnodal singular points at P_1, P_2, P_3, P_4 and satisfying the property $\mathcal{P}$ of Theorem 2.1, where P_1, P_2, P_3, P_4 are the points defined in Proposition 3.3.

Let $\mathcal{S}_{5,P}$ be the space of such normal quintic surfaces F_5, where we denote P for four points P_1, P_2, P_3, P_4. Then for a general quintic surface F to have tacnodal singularities at P_1, P_2, P_3, P_4, we need $4 \times 10 = 40$ conditions from Proposition 3.3. It is not difficult to find out that we need two 3 more conditions for two tacnodal planes to be identical, thus total 6 more conditions. Then $\dim \mathcal{S}_{5,P} = 55 - 4 \times 10 - 6 = 9$, where 55 is the projective dimension of the space of homogeneous quintic polynomials.

It is clear that any other normal quintic surface in $\mathbf{P}^3$ with the property $\mathcal{P}$ at different four points in general position is the linear transform of a normal quintic surface $F_5 \in \mathcal{S}_{5,P}$. Thus every normal quintic surface with the property $\mathcal{P}$ at any four points in general position belongs to an open subset of $\mathcal{S}_{5,P} \times \mathbb{C}^{16}$, where $\mathbb{C}^{16} = \mathbb{C}^4 \times \cdots \times \mathbb{C}^4$ represents the space of four points in $\mathbf{P}^3$. There is a diagonal action on $\mathcal{S}_{5,P} \times \mathbb{C}^{16}$ by the group $\mathbf{T} \times GL(4)$ with the torus group $\mathbf{T}$. Hence we have the birational isomorphism

$$\mathcal{Q} \simeq (\mathcal{S}_{5,P} \times \mathbb{C}^{16})/(\mathbf{T} \times GL(4))$$

From this birational isomorphism, $\dim\mathcal{Q} = (9+16) - (3+16) = 6$. □

COROLLARY 3.5. *Let $\mathcal{E}$ be the moduli space of Enriques surfaces which are the minimal models of normal quintic surfaces in $\mathbf{P}^3$ satisfying the property $\mathcal{P}$ of Theorem 2.1. Then dim $\mathcal{E}$ is 6.*

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