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ON PRIME DUAL IDEALS IN BCK-ALGEBRAS

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ABSTRACT. In [1], Ahmad has given a characterization of prime dual ideals in bounded commutative BCK-algebras. The aim of this paper is to show that Theorem of [1] holds without the commutativity.

By a BCK-algebra we mean an algebra $(X;*,0)$ of type (2,0) satisfying the following axioms:

(a_1) $(x*y)*(x*z) \leq (z*y)$,
(a_2) $x*(x*y) \leq y$,
(a_3) $x \leq x$,
(a_4) $x \leq y$ and $y \leq x$ implies $x = y$,
(a_5) $0 \leq x$,

where $x \leq y$ is defined by $x * y = 0$.

From now on, for any BCK-algebra X, $\leq$ is called a BCK-ordering on X and we know that $(X, \leq)$ is a partially ordered set. For any BCK-algebra X, the set $\{x \in X | x \leq a\}$ is denoted by $A(a)$, where a is a fixed element in X. A BCK-algebra X is said to be with condition (S) if there is a largest element x satisfying $x*a \leq b$ for any two fixed elements $a, b \in X$. The largest element is denoted by $a \circ b$. A nonempty subset A of a BCK-algebra X is called an ideal of X if it satisfies $0 \in A$, and $x*y, y \in A$ imply $x \in A$ for all $x, y \in X$.

A BCK-algebra X is said to be bounded if there exists $1 \in X$ such that $x \leq 1$ for all $x \in X$. In a bounded BCK-algebra, we denote $1 * x$ by Nx. A BCK-algebra X is said to be commutative if it satisfies for all $x, y \in X$, $x*(x*y) = y*(y*x)$. A BCK-algebra X is said to be positive implicative if $(x*y)*z = (x*z)*(y*z)$ for all $x, y, z \in X$.

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A partially ordered set $(L, \leq)$ is called a lower semilattice if every pair of elements in L has a greatest lower bound (meet); it is called an upper semilattice if every pair of elements in L has a least upper bound (join); the operations join and meet are denoted by $\vee$ and $\wedge$, respectively. If $(L, \leq)$ is both an upper and a lower semilattice, then it is called a lattice.

DEFINITION 1. A nonempty subset D of a BCK-algebra X is said to be a dual ideal of X if

(a_6) $a \in D$ and $a \leq b$ imply $b \in D$,
(a_7) $a, b \in D$ imply there exists an element $c \in D$ such that $c \leq a$ and $c \leq b$.

Clearly, a principal dual ideal $D(a)$ generated by a is $\{x \in X | a \leq x\}$ (see [2]).

DEFINITION 2. A dual ideal D in a BCK-algebra X is called a prime dual ideal if for any $a, b \in X$, $a \vee b \in D$ implies $a \in D$ or $b \in D$ where $a \vee b = lub\{a, b\}$.

For any subsets A, B of an upper semilattice BCK-algebra X, we define

$$A \vee B = \{x \vee y | x \in A, y \in B\},$$

where $x \vee y = lub\{x, y\}$.

It is well known that if X is a bounded commutative BCK-algebra, then $x \vee y = N(Nx \wedge Ny)$.

B. Ahmad [1] proved the following theorem.

THEOREM 3. *Let X be a bounded commutative BCK-algebra. Then the following are equivalent:*

(b_1) *D is a prime dual ideal of X,*
(b_2) *For any dual ideals D_1, D_2 of X, $D_1 \vee D_2 \subset D$ implies $D_1 \subset D$ or $D_2 \subset D$.*

Now we prove that Theorem 3 holds without the commutativity.

THEOREM 4. *Let X be an upper semilattice BCK-algebra. Then the conditions (b_1) and (b_2) are equivalent.*

PROOF. Suppose that D is a prime dual ideal such that $D_1 \vee D_2 \subset D$ where D_1 and D_2 are dual ideals of X. In order to prove that $D_1 \subset D$ or $D_2 \subset D$, let us assume the contrary that neither $D_1 \subset D$ nor $D_2 \subset D$. Then there exist $a \in D_1, b \in D_2$ such that $a \notin D$ and $b \notin D$. Since $a \vee b \in D_1 \vee D_2$ and $D_1 \vee D_2 \subset D$, we have $a \vee b \in D$. D being prime implies that $a \in D$ or $b \in D$, which is a contradiction.

Conversely, suppose that for any dual ideals D_1, D_2 of X, $D_1 \vee D_2 \subset D$ implies $D_1 \subset D$ or $D_2 \subset D$. We claim that D is a prime dual ideal. Let $a, b \in X$ be such that $a \vee b \in D$. Note that

$$D(a) = \{x \in X | a \leq x\} \text{ and } D(b) = \{x \in X | b \leq x\}$$

are principal dual ideals generated by a and b, respectively. It is sufficient to show that $D(a) \vee D(b) \subset D$. Let $x \in D(a)$ and $y \in D(b)$. Then we have $a \leq x$ and $b \leq y$ and hence $a, b \leq x \vee y$. Thus $a \vee b \leq x \vee y$. It follows from (a_6) that $x \vee y \in D$. Therefore $D(a) \vee D(b) \subset D$. Then by our assumption $D(a) \subset D$ or $D(b) \subset D$, so that in particular $a \in D$ or $b \in D$ which proves that D is a prime dual ideal. This completes the proof.

REMARK. We note that the results of Theorem 4 is capable of generalization to the case of lattices.

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