Comm. Korean Math. Soc. 10 (1995), No. 3, pp. 519–525

ON COHOMOLOGY GROUPS OF $F_p[t]$-MODULE SCHEMES

SUNG SIK WOO

ABSTRACT. By using an exact sequence of extension groups corresponding to an isogeny of a Drinfeld module we investigate which extension classes are coming from Hom(G,C). In the last section of this paper an example was given where the connecting homomorphism can be explictly computed.

§1. Introduction

In [W2] it was shown that a certain subgroup of extension group of a rank 2 Drinfeld module by the Carlitz module is a good candidate for a dual of a rank 2 Drinfeld module which is also a good analogy of the dual abelian variety. In case of rank bigger than two, the situation appears to be much more complicated. In case of rank 2, the idea of choosing the subgroup was finding the smallest subgroup containing all the image of the connecting homomorphisms

$$\delta_\phi \colon Hom(G,C) \to Ext(E,C)$$

for all isogeny ϕ, where G is the kernel of ϕ. In this paper we investigate the map δ_ϕ and compute some explict examples.

Received December 8, 1994. Revised April 10, 1995.
1991 AMS Subject Classification: 11G18.
Key words: t-module, Drinfeld module, extension group.

§2. $\mathbb{F}_p[t]$-module schemes

Throughout this paper we fix the following notations: p is a fixed prime, A is the polynomial ring $\mathbb{F}_p[t]$, K is a perfect field containing A and T is the image of t in K. As usual $\mathbb{G}_{a,K}$ denotes the additive group scheme over K. It is well known that the ring of endomorphisms $End_K(\mathbb{G}_a)$ is a noncommutative polynomial ring $K[\tau]$ with a commutation relation,

$$\tau x = x^p \tau \quad \text{for } x \in K.$$

An elliptic module or a Drinfield module E of rank r is the $\mathbb{F}_p[t]$-module scheme $\mathbb{G}_a$ with an A-action

$$\psi : A \to End_K(\mathbb{G}_a) = K[\tau]$$

such that

(i) degree of ψ_a in τ is the same as deg(a)r,
(ii) the constant term of ψ_a is the same as the image of a in K.

If (E_1, ψ_1) and (E_2, ψ_2) are elliptic modules then an isogeny from E_1 to E_2 is defined to be an endomorphism u of $\mathbb{G}_a$ such that $u \circ \psi_1 = \psi_2 \circ u$.

Anderson [A] gave a definition of higher dimensional analogue of Drinfeld modules: An abelian t-module over K is an A-module valued functor E such that

(i) as a group valued functor, E is isomophic to $\mathbb{G}_a^n$ for some n,
(ii) $(t - T)^N Lie(E) = 0$ for some positive integer N,
(iii) there is a finite dimensional subspace V of the group $Hom(E, \mathbb{G}_a)$ of the morphisms of K-algebraic groups such that

$$Hom(E, \mathbb{G}_a) = \sum_{j=0}^{\infty} V \circ t^j.$$

A morphism between t-modules is simply a natural transformation of the functors.

Let $K[t, \tau]$ be the noncommutative ring generated by t and τ over K with the relations; $t\tau = \tau t$, $xt = tx$, $\tau x = x^p \tau$ for $x \in K$. A t-motive M is a left $K[t, \tau]$-module with the following properties,

(i) M is free of finite rank over K[t],
(ii) $(t - T)^N (M/\tau M) = 0$ for some positive integer N,
(iii) M is finitely generated over $K[\tau]$-linear map.

A morphism between t-motives is simply a $K[t,\tau]$-linear map. For a t-module E let M(E) be the set of all morphisms $E \to \mathbb{G}_a$ of K-algebraic groups equipped with $K[t,\tau]$-module structure,

$$(xm)(e) = x(m(e)),$$
$$\tau(m)(e) = m(e)^p,$$
$$tm(e) = m(t(e)),$$

for $e \in E$. Anderson [A] proved:

THEOREM 1. *The functor sending E to M(E) is an anti-equivalence of categories between t-modules and t-motives.*

In particular, we have a canonical isomorphism

$$Ext_{t-module}(E,C) \longrightarrow Ext_{K[t,\tau]}(M(C),M(E)).$$

In [W1] we had,

THEOREM 2. *Let E be a Drinfeld module of of rank r. Then the group Ext(E,C) is isomorphic to K^r as additive groups and is represented by $K[\tau]/\mathcal{B}$ where $\mathcal{B} = \{\alpha\psi_t^E - \psi_t^C\alpha | \alpha \in K[\tau]\}$.*

Another result we want to recall from [W1] is:

THEOREM 3. *Let E be a Drinfeld module of rank r. Let ϕ be an isogeny of E and let $G = Ker(\phi)$. Then we have an exact sequence,*

$$0 \to Hom(G,C) \xrightarrow{\delta_\phi} Ext(E,C) \xrightarrow{\phi^*} Ext(E,C),$$

For $f \in Hom(G,C)$, δ_ϕ is given by the formula

$$\bar{f}\psi_t^E - \psi_t^C\bar{f} = \delta_\phi(f)\phi$$

where $\bar{f}$ is a lift of f in $K[\tau]$.

§3. The connecting homomorphism δ_ϕ.

First we start by proving t-linearity of δ_ϕ.

PROPOSOTION. *The connecting homomorphism δ_ϕ is t-linear.*

PROOF. The result follows from the general theory of homological algebra. However, we provide a proof because it will give some insight into the nature of the connecting homomorphism. In [W1] we had the formula,

$$\delta_\phi(f)\phi = f\psi_t^E - \psi_t^C(f),$$

Hence we have

$$\delta_\phi(t \cdot f)\phi = (t \cdot f)\psi_t^E - \psi_t^C(t \cdot f),$$

where $t \cdot f$ denotes the t-action on f. Now the right hand side becomes

$$(f\psi_t^E - \psi_t^C f)\psi_t^E = \delta_\phi(f)\phi\psi_t^E = (\delta_\phi(f)\psi_t^E\phi = (t \cdot \delta_\phi(f))\phi,$$

which proves t-linearity of δ_ϕ.

Let ϕ be an isogeny of a Drinfeld module E and δ_ϕ be the corresponding connecting homomorphism. (Compare the following with [S] Ch.VII.)

THEOREM 1. *Identify the group Ext(E,C) with $K[\tau]/\mathcal{B}$ as before. Let $f \in Ext(E, C)$ and let*

$$0 \to C \to \mathcal{E}_f \xrightarrow{\pi_f} E \to 0$$

be the corresponding extension. Then $\mathcal{E}_f$ belongs to the image of δ_ϕ for some isogeny ϕ if and only if π_f lifts the isogeny ϕ, i.e., there is $\Phi \colon E \to \mathcal{E}_f$ such that $\pi \circ \Phi = \phi$.

PROOF. First suppose $\mathcal{E}_f$ belongs to $Im(\delta_\phi)$. Then $\phi^*\mathcal{E}_f$ is a trivial extension. Hence there is a section $s \colon E \to \phi^*\mathcal{E}_f$ which makes the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C & \longrightarrow & \phi^*\mathcal{E}_f & \xleftarrow{\;s\;} & E & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle\beta} & & \downarrow & & \\ 0 & \longrightarrow & C & \longrightarrow & \mathcal{E}_f & \longrightarrow & E & \longrightarrow & 0 \end{array}$$

commutative. Now define Φ by $\Phi = \beta \circ s$.

Conversely assume that $\pi_f : \mathcal{E}_f \to E$ lifts an isogeny ϕ of E to $\Phi : E \to \mathcal{E}_f$. By using Theorem 1 of §2, we compute in the category of t-motives: Since we already know that $\mathcal{E}_f \cong \mathbb{G}_a^2$ we can write

$$\Phi(x, y) = \Phi((x, 0) + (0, y)) = \Phi(x) + \Phi(0, y)$$

where $\Phi : M(\mathcal{E}_f) \to M(E)$. Set $\varepsilon(y) = \Phi(0, y)$. Using the description of t-action on $\mathcal{E}_f$ in [W1], we have

$$t \cdot (0, y) = (yf, y\psi_t^C).$$

We compute

$$\begin{aligned} t_E \cdot \varepsilon(y) &= \Phi(yf, y\psi_t^C) = \Phi((yf, 0) + (0, y\psi_t^C)) \\ &= \phi(y, f) + \varepsilon(y\psi_t^C) \\ &= yf\phi + \varepsilon(t_C \cdot y). \end{aligned}$$

Hence

$$\varepsilon\psi_t^E - \psi_t^C \varepsilon = f\phi.$$

Therefore $\delta_\phi(\varepsilon) = f$ as desired.

Given an extension $\mathcal{E}_f$ corresponding to $f \in Ext(E, C) = K[\tau]/\mathcal{B}$ we know that $\mathcal{E}_f$ is isomorphic to $\mathbb{G}_a^2$ as group schemes. Hence there is a natural section $s : E \to \mathcal{E}_f$

$$0 \to C \to \mathcal{E}_f \underset{\pi_f}{\overset{s}{\leftrightarrows}} E \to 0$$

sending x to $(x, 0)$. If $\mathcal{E}_f$ is in the image of δ_ϕ for some ϕ then the section s is not far from being a t-morphism:

THEOREM 2. *Using the notations above if $\mathcal{E}_f$ is in the image of δ_ϕ for some isogeny ϕ then we have*

$$ts(x) - st(x) = (0, \bar{f}\psi_t^E(a) - \psi_t^C \bar{f}(a))$$

where $\bar{f}$ is a lift of f in $K[\tau]$ and $\phi(a) = x$.

PROOF. Let $G = Ker(\phi)$. We know that $\mathcal{E}_f$ is isomorphic to $E \times C/D_f$ where $D_f = \{(-a', f(a'))|a' \in G\}$. We identify $\mathcal{E}_f$ with $E \times C$ by the map

$$\beta \colon E \times C/D_f \to E \times C$$

which is defined by $\beta(a,b) = (\phi(a), b + \bar{f}(a))$. We let t act on $E \times C/D_f$ and send it to $E \times C$ by β;

$$t \cdot s(x) = (\psi_t^E(x), \bar{f}\psi_t^E(a) - \psi_t^C \bar{f}(a))$$

where $\phi(x) = x$. On the other hand, we have

$$s(t \cdot x) = (\psi_t^E(x), 0),$$

as desired.

§4 An example

Before we take our example we prove an easy fact first:

PROPOSITION 1. *Let $f \in K[\tau]$ and let ϕ be an isogeny of a Drinfeld module E, and let $G = Ker(\phi)$. Then $f|G$ is a t-morphism if and only if there is $\delta(f)$ such that $f\psi_t^E - \psi_t^C f = \delta(f)\phi$.*

PROOF. If $f|G$ is a t-morphism then $f\psi_t^E - \psi_t^C f = 0$ on G. Hence it factors through ϕ. Conversely if the above equation is satisfied then by restricting both sides to G we see that the right hand side becomes zero. Hence $f|G$ is a t-morphism.

For our example we take $\psi_t^E = \tau^3 + T$ and $\psi_t^C = \tau + T$. Let

$$a = x_n t^n + x_{n-1}t^{n-1} + \cdots + x_0 \in \mathbb{F}_p[t] \ (x_n \neq 0)$$

and choose $\phi = \psi_a^E$. In this case we have a nice description for the connecting homomorphism:

PROPOSITION 2. *For $f \in K[\tau]/K[\tau]\psi_a^E$ we have*

$$\delta_\phi(a_{3n-1}\tau^{3n-1} + a_{3n-2}\tau^{3n-2} + \dots) = x_n{}^{-1}(a_{3n-1}\tau^2 + a_{3n-2}\tau).$$

PROOF. In the formula $f\psi_t^E - \psi_t^C f = \delta(f)\phi$, we know that $\delta(f)$ is of the form $\alpha\tau^2 + \beta\tau$ with degree of f is 3n-1. Writing out we get,

$$\begin{aligned}&(a_{3n-1}\tau^{3n-1} + a_{3n-2}\tau^{3n-2} + \dots)(\tau^3 + T)\\&- (\tau + T)(a_{3n-1}\tau^{3n-1} + a_{3n-2}\tau^{3n-2} + \dots)\\=&(\alpha\tau^2 + \beta\tau)(x_n\tau^{3n} + \dots).\end{aligned}$$

To complete the proof compare the coefficients of degree τ^{3n+2} and τ^{3n+1}.

References

[A] G. Anderson, *t-motives*, Duke Math. Jour. **53(2)** (1986).
[S] J. P. Serre, *Algebraic groups and class fields*, G. T. M. No. 117, Springer-Verlag, 1988.
[W1] S. S. Woo, *Extensions of t-modules*, Comm. K. M. S. **9** (1994).
[W2] S. S. Woo, *Extensions of Drinfeld modules of rank 2 by the Carlitz module*, to appear in J. of K. M. S. **33**.

Department of Mathematics
Ewha Women's University
Seoul 120-750, Korea.