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ON THE CONSTRUCTION AND THE EXISTENCE OF PARAMETRIC CUBIC g^2 B-SPLINE

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ABSTRACT. A parametric cubic spline interpolating at fixed number of nodes is constructed by formulating a parametric cubic g^2 B-splines $S_3(t)$ with not equally spaced parametric knots. Since the fact that each component is in C^2 class is not enough to provide the geometric smoothness of parametric curves, the existence of $S_3(t)$ oriented toward the modified second-order geometric continuity is focalized in our work.

1. Introduction

The cubic splines are widely used to solve the problem on interactive designing of the free-form curves because they are the "smoothest" functions in the sense of Holladay's minium curvature property theorem[1, pp75-77]. It is also the natural desire to adapt the parametric form when a curve is defined by arbitrary points. The parametric form is axis-independent, and can also represent the closed or multi-valued curves. Each component is periodic if the curve is closed. The self-intersective curve has nonself-intersective functions as its component, since the ordered sequence of parameters is monotonically increasing. A variance of a single node can affect the whole curve if the approach is based upon an interpolating curve which passes through the whole given nodes. But it has been known that a single node has a local effect in the piecewise interpolating approach.

This paper is concerned with the parametric cubic spline curves which must interpolate a fixed number of points $p_i = (x_i, y_i)$: the p_i's are the

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nodes of a curve. For simplicity we consider curves in R^2, but everything extends easily to R^3. The *formula of (parametric) cubic g^2 B-splines with not equally spaced (parametric) knots* is constructed in section 2. Since the fact that each component is in C^2 class is not enough to provide the geometric smoothness of parametric curves, this paper will be focused on *the existence of (parametric) cubic g^2 B-splines oriented toward the modified second-order geometric continuity* in section 3 which was introduced by Barsky[2] and developed by Nielson[6] and Goodman[4][5].

2. The construction of parametric cubic g^2 B-splines

A parametric curve $C_3(t) = (x(t), y(t))$ on the interval$[t_1, t_n]$ is said to be in C^k provided each component function $x(t), y(t)$ has a continuous k-th derivative on the interval $[t_1, t_n]$. However, this type of continuity are unnecessarily restrictive when the graph of a parametric curve must be geometrically smooth and continuous. To overcome this gap, Barsky introduced the definition of first-order geometric continuity, G^1, if each component $x(t), y(t)$ of $C_3(t)$ is continuous and if the unit tangent vector $T(t)$ to the curve is also continuous, and of second-order geometric continuity, G^2, if $C_3(t)$ is in G^1 and if its curvature vector $K(t)$ is continuous. Goodman[4] generalized this geometric continuity to G^k. For the mathematical notation of these geometric continuity in R^2, the following two definitions are modified using the inner product form.

DEFINITION 2.1. The class $G^2[t_1, t_n]$ is defined by the set of a parametric curve $C_3(t)$ satisfying the following conditions :

i) $C_3(t) \in C^0[t_1, t_n]$,

ii) $T(t) = \dot{C}_3(t)/|\dot{C}_3(t)| \in C^0[t_1, t_n]$,

iii) $K(t) = \{\ddot{C}_3(t) - T(t)[\ddot{C}_3(t) \cdot T(t)]\}/|\dot{C}_3(t)|^2 \in C^0[t_1, t_n]$.

LEMMA 2.1. *If a curve $C_3(t)$ is in G^2, then $C_3(a(u))$ is in C^2, where $a(u)$ is the inverse (that is, $a(p(t)) = t$) of arclength:*

$$p(t) = \int_{t_1}^{t} |C_3'(s)|ds, \quad t_1 \le t \le t_n.$$

PROOF. It can be conferred in [6].

Since $C_3(a(u))$ and $C_3(t)$ have the same graph, the curves of G^2 are just as continuous as those of C^2; at least visually. But the inverse of Lemma 2.1 is not true(cf.[2, pp21-25]). The parametric derivative vectors do not provided an appropriate measure of geometric continuity. This paper is concerned mainly with piecewise parametric cubic curves, and we modify a slightly different continuity class, g^2.

DEFINITION 2.2. The class $g^2[t_1, t_n]$ is defined by the set of a parametric curve $C_3(t)$ satisfying the following conditions :

i) $C_3(t) \in C^0[t_1, t_n]$,

ii) $\dot{C}_3(t) \in C^0[t_1, t_n]$,

iii) $K(t) = \{\ddot{C}_3(t) - T(t)[\ddot{C}_3(t) \cdot T(t)]\}/|\dot{C}_3(t)|^2 \in C^0[t_1, t_n]$

with same $T(t)$ of DEFINITION 2.1.

It is clear that $g^2[t_1, t_n]$ is contained in $G^2[t_1, t_n]$. It is also interesting that these two definitions of continuity are equivalent for piecewise polynomial curves by the following Nielson's theorem.

THEOREM 2.1. (Nielson's theorem) *For every parametric piecewise polynomial curve in $G^2[t_1, t_n]$, there is a parametric piecewise polynomial curve in $g^2[t_1, t_n]$ with the same graph.*

PROOF. It can be proved in [6] using a homeomorphic piecewise linear function(cf. [3]) for the change of parameter.

For $i = 0, 1, ..., N-1$, let $p_i = (x_i, y_i)$ be points in R^2 which are referred to as interpolating nodes. The problem is to construct a smooth curve $S_3(t) = (x_3(t), y_3(t))$ called the *parametric cubic g^2 B-spline.* We assume that the smooth curve passes all nodes in the given order and $S_3(t) = 0$ at two end nodes.

The parameteric interval $[a, b]$ is partitioned in such a way that $a = t_0 < t_1 < ... < t_{N-1} = b$ since the curve interpolate N nodes. Whether the nodes are closed to being evenly spaced or not, it is advisable [1, p51] to take the *cumulative chord length*, i.e.,

$$t_0 = a,$$
$$t_i = d[(x_i, y_i), (x_{i-1}, y_{i-1})] + t_{i-1}, \qquad i = 1, 2, ..., N-1,$$

where $d[...]$ is Euclidean distance in the plane. By the normalization process,the parametric interval $[a, b]$ can be changed to $[0, 1]$ for the numerical computation, i.e.,

$$u_0 = t_0 - a = 0,$$
$$u_i = (t_i - a)/(b - a), \qquad i = 1, 2, ..., N-1.$$

This parametrization results the sequence of pairs $\{(t_i, x_i)\}, \{(t_i, y_i)\}$, $i = 0, 1, ..., N-1$ from the sequence of points $\{(x_i, y_i)\}, i = 0, 1, ..., N-1$.

To get, $x_3(t)$, x component of the parametric cubic g^2 B-spline $S_3(t)$, the interval $[t_0, t_{N-1}]$ is partitioned subintervals $\{I_i\}, i = -1, 0, 1, ..., N-1, N$ by

$$\begin{aligned} I_i &= [t_{i-2}, t_{i-1}, t_i, t_{i+1}, t_{i+2}], \quad i = 2, 3, ..., N-3, \\ I_{-1} &= [t_0, t_0, t_0, t_0, t_1], \qquad I_N = [t_{N-2}, t_{N-1}, t_{N-1}, t_{N-1}, t_{N-1}], \\ I_0 &= [t_0, t_0, t_0, t_1, t_2], \qquad I_{N-1} = [t_{N-3}, t_{N-2}, t_{N-1}, t_{N-1}, t_{N-1}], \\ I_1 &= [t_0, t_0, t_1, t_2, t_3], \qquad I_{N-2} = [t_{N-4}, t_{N-3}, t_{N-2}, t_{N-1}, t_{N-1}]. \end{aligned}$$

For each subinterval I_i, let the cardinal $B_i(t)$ be defined by

$$B_i(t) = \begin{cases} L^+_{i+1,i-2}(t) \cdot L^+_{i,i-2}(t) \cdot L^+_{i-1,i-2}(t), & \text{for } t_{i-2} \le t \le t_{i-1} \\ L^+_{i+1,i-2}(t) \cdot L^+_{i,i-2}(t) \cdot L^-_{i,i-1}(t) & \\ \quad + L^+_{i+1,i-2}(t) \cdot L^-_{i+1,i-1}(t) \cdot L^+_{i,i-1}(t) & \\ \quad + L^-_{i+2,i-1}(t) \cdot L^+_{i+1,i-1}(t) \cdot L^+_{i,i-1}(t), & \text{for } t_{i-1} \le t \le t_i \\ L^+_{i+1,i-2}(t) \cdot L^-_{i+1,i-1}(t) \cdot L^-_{i+1,i}(t) & \\ \quad + L^-_{i+2,i-1}(t) \cdot L^+_{i+1,i-1}(t) \cdot L^-_{i+1,i}(t) & \\ \quad + L^-_{i+2,i-1}(t) \cdot L^-_{i+2,i}(t) \cdot L^+_{i+1,i}(t), & \text{for } t_i \le t \le t_{i+1} \\ L^-_{i+2,i-1}(t) \cdot L^-_{i+2,i}(t) \cdot L^-_{i+2,i+1}(t), & \text{for } t_{i+1} \le t \le t_{i+2} \\ 0, & \text{otherwise} \end{cases}$$

where $L^+_{j,k}(t), L^-_{j,k}(t)$ are defined by

$$L^+_{j,k}(t) = \begin{cases} (t-t_k)/(t_j-t_k), & \text{for } j > k \\ 0, & \text{for } j \le k \end{cases}$$

and

$$L^-_{j,k}(t) = \begin{cases} (t_j-t)/(t_j-t_k), & \text{for } j > k \\ 0, & \text{for } j \le k. \end{cases}$$

Then, the x component of $S_3(t)$ is

$$x_3(t) = \sum_{i=-1}^{N} a_i B_i(t) \tag{2.1}$$

and the a_i's can be determined by the interpolating constraints $x_3(t_i) = x_i$, $i = 0, ..., N-1$ and two end conditions $x_3'(t_0) = 0 = x_3'(t_{N-1})$[3].

Similarly, the y component of $S_3(t)$ is

$$y_3(t) = \sum_{i=-1}^{N} b_i B_i(t) \tag{2.2}$$

and the b_i's can be also determined by $y_3(t_i) = y_i, i = 0, 1, ..., N-1$ and $y_3'(t_0) = 0 = y_3'(t_{N-1})$.

REMARK. If the curve is closed, the subintervals are changed by periodic property[5] as following;

$$I_{-1} = [t_{N-3}, t_{N-2}, t_{N-1}, t_0, t_1] = I_{N-1},$$
$$I_0 = [t_{N-2}, t_{N-1}, t_0, t_1, t_2] = I_N,$$
$$I_1 = [t_{N-1}, t_0, t_1, t_2, t_3], \qquad I_{N-2} = [t_{N-3}, t_{N-2}, t_{N-1}, t_0, t_1]$$

and each component of the parametric cubic g^2 B-spline is

$$x_3(t) = \sum_{i=-1}^{N} a_i B_i(t) \qquad y_3(t) = \sum_{i=-1}^{N} b_i B_i(t) \tag{2.3}$$

where the coefficient a_i's and b_i's can be determined only by the interpolating constraints $x_3(t) = x_i$, $y_3(t_i) = y_i, i = 0, 1, ..., N-1$.

3. The existence of parametric cubic g^2 B-splines

The existence and uniqueness of parametric cubic g^2 B-spline $S_3(t)$ which is constructed by (2.1) and (2.2) or (2.3), are verified as follows.

THEOREM 3.1. *There exists a unique cubic g^2 B-spline $S_3(t)$ in $C^2[t_0, t_{N-1}]$ satisfying the interpolating constraints:*

$$S_3'(t_0) = f'(t_0), \quad S_3'(t_{N-1}) = f'(t_{N-1}),$$

$$S_3(t_i) = f(t_i), \quad \text{for } i = 0, 1, ..., N-1,$$

when the function $f(t)$ is defined by the set of N points $\{(t_i), f(t_i)\}$, for a distinct increasing sequence $t_0 < t_1 < .. < t_{N-1}$.

PROOF. By the formula of $B_i(t)$ in section 2, simple algebraic calculations show that $B_i(t)$ has continuous first-order derivative in the open interval (t_{i-1}, t_{i+2}) as below:

$$B_i(t_j) = \begin{cases} (t_{i-1} - t_{i-2})^2/(t_{i+1} - t_{i-2})(t_i - t_{i-2}), & \text{for } j = i-1 \\ (t_i - t_{i-2})(t_{i+1} - t_i)/(t_{i+1} - t_{i-2})(t_{i+1} - t_{i-1}) & \\ \quad +(t_{i+2} - t_i)(t_i - t_{i-1})/(t_{i+2} - t_{i-1})(t_{i+1} - t_i), & \text{for } j = i \\ (t_{i+2} - t_{i+1})^2/(t_{i+2} - t_{i-1})(t_{i+2} - t_i), & \text{for } j = i+1 \\ 0, \quad \text{for } j = i-2 \text{ or } j = i+2 & \end{cases}$$

$$B_i'(t_j) = \begin{cases} 3(t_{i-1} - t_{i-2})/(t_{i+1} - t_{i-2})(t_i - t_{i-2}), & \text{for } j = i-1 \\ 3\{(t_{i+1} - t_i)(t_{i+2} - t_i) - (t_i - t_{i-1})(t_i - t_{i-2})\}/ & \\ \quad (t_{i+1} - t_{i-2})(t_{i+2} - t_{i-1})(t_{i+1} - t_{i-1}), & \text{for } j = i \\ -3(t_{i+2} - t_{i+1})/(t_{i+2} - t_{i-1})(t_{i+2} - t_i), & \text{for } j = i+1 \\ 0, \quad \text{for } j = i-2 \text{ or } j = i+2 & \end{cases}$$

$$B_i''(t_j) = \begin{cases} 6/(t_{i+1} - t_{i-2})(t_i - t_{i-2}), & \text{for } j = i-1 \\ -6/(t_{i+1} - t_{i-1})\{1/(t_{i+1} - t_{i-2}) + 1/(t_{i+2} - t_{i-1})\}, & \\ \quad \text{for} j = i & \\ -6/(t_{i+2} - t_{i-1})(t_{i+2} - t_i), & \text{for } j = i+1 \\ 0, \quad \text{for } j = i-2 \text{or } j = i+2. & \end{cases}$$

Let $s_3(t) = \sum_{i=-1}^{N} c_i B_i(t)$, then the interpolating constraints determine each coefficient c_i from the linear system:

$$[a_{ij}][c_i] = [f'(t_0), f(t_0), ..., f(t_{N-1}), f'(t_{N-1})]^T, \quad i, j = -1, 0, ..., N,$$

where $a_{-1j} = B'_{-1}(t_j)$, $a_{Nj} = B'_{N-1}(t_j)$ and $a_{ij} = B_i(t_j)$ for $i, j = 0, 1, ..., N-1$. Since the matrix $[a_{ij}]$ is diagonally dominant with strict diagonal dominance on all but first and last rows, it follows from Gershgorin's theorem that $[a_{ij}]$ is nonsingular. Hence the linear system has a unique solution $[c_i](i = -1, 0, ..., N)$.

THEOREM 3.2. *There exists a unique parametric cubic g^2 B-spline $S_3(t) = (x_3(t), y_3(t))$ in $C^2[t_0, t_{N-1}]$, whose components $x_3(t), y_3(t)$ are the same cubic g^2 B-spline in Theorem 3.1 when the arbitrary N points $(x_i, y_i), i = 0, 1, ..., N-1$ are parametrized to $((t_i, x_i), (t_i, y_i)), i = 0, 1, ..., N-1$ by the cumulative chord length.*

PROOF. By virtue of Theorem 3.1, the existence and uniqueness of each component $x_3(t), y_3(t)$ yields directly both of the parametric cubic g^2 B-spline $S_3(t)$.

THEOREM 3.3. *The parametric cubic g^2 B-spline $S_3(t)$ in Theorem 3.2 is in $G^2[t_0, t_{N-1}]$.*

PROOF. Since $S_3(t)$ is in C^2 class and the tangent vector $\dot{S}_3(t)$ is not equal to zero vector at each nodes $t_i, i = 0, 1, ..., N-1$, the condition i) and ii) of DEFINITION 2.2 are obviously true. So, whether the inner product $S_3(t) \cdot T(t)$ is constant or zero, the curvature vector $K(t)$ is continuous at each knots $t_i, i = 0, 1, ..., N-1$ because $T(t)$ and $(x''_3(t), y''_3(t))$ are continuous in $[t_0, t_{N-1}]$. Hence $S_3(t)$ is in G^2 class.

THEOREM 3.4. *There exists a parametric cubic g^2 B-spline $S_3(t)$ in $g^2[t_0, t_{N-1}]$ where t_i's are reparametrized piecewisely into $s_0 = t_0 < s_1 < ... < s_{N-1} = t_{N-1}$ and $S_3(t)$ is the same in Theorem 3.2.*

PROOF. Since $S_3(t)$ is also a parametric piecewise cubic polynomial curve and is in $G^2[t_0, t_{N-1}]$, Nielson's Theorem implies that there exist an appropriate homeomorphism h of $[t_0, t_{N-1}]$ onto itself which $S_3(h(t)) = S_3(t)$ is in $g^2[t_0, t_{N-1}]$.

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