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UPPER AND LOWER BOUNDS FOR ANISOTROPIC TORSIONAL RIGIDITY

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ABSTRACT. Some bounds for anisotropic torsional rigidity with one plane of elastic symmetry perpendicular to the axis of the beam are derived by making use of the isoperimetric inequalities, complementary variational principles, and the maximum principle. Upper and lower bounds are obtained by applying the isoperimetric inequalities. while the upper bound investigated by the variational principles and maximum principle. The analysis is patterned after the work of Payne and Weinberger [J. Math. Anal. Appl. 2(1961), pp.210-216].

1. Introduction

In this present work we would like to bound the torsional rigidity of a homogeneous anisotropic elastic beam with one plane of elastic symmetry perpendicular to the axis of the beam in terms of the only geometrical quantities which are the perimeter, area and second moments of the cross-section by using the isoperimetric inequality. We show that lower and upper bounds are easily obtained by employing Schwarz's inequality, and the weighted arithmetic-geometric mean inequality. An application of a suitable affine transformation to an anisotropic boundary value problem is reduced to an equivalent isotropic boundary problem for the transformed cross-section; further, the torsional rigidity of the anisotropic problem is observed to be equivalent to the torsional rigidity of the isotropic beam with the transformed cross-section. In Section 2 we investigate upper and lower bounds for rigidity using the classical isoperimetric inequality. The lower bound for the anisotropic torsional

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rigidity of this paper represents an extension to the work of Payne and Weinberger [7], while the upper bound represents a sharpened version of the work of Payne [4] by the variational principles in Section 3. In Section 4, we also derive an elliptic inequality ensuring the maximum principle which will yield an upper bound.

2. Upper and lower bounds using the isoperimetric inequality

Let us consider an elastic cylinder of arbitrary simply-connected cross-section R whose longitudinal axis coincides with the z-axis of a Cartesian rectangular system x, y, and z. The material of the cylinder displays a general rectilinear anisotropy subject to the condition that at each point of the body the planes parallel to the xy plane constitute planes of elastic symmetry. As in the isotropic case the present anisotropic torsion problem may be formulated in terms of Prandtl's stress function $\varphi(x,y)$ as follows (see [2])

$$
\begin{aligned} a\varphi_{,xx} - 2b\varphi_{,xy} + c\varphi_{,yy} &= -2 \quad \text{in} \quad R, \\ \varphi &= 0 \quad \text{on} \quad \partial R, \end{aligned} \tag{2.1}
$$

where a, b and c are constants of the elastic material and a comma denotes differentiation. To reduce the anisotropic boundary value problem to the equivalent isotropic boundary value problem, we introduce the following affine transformation

$$
\xi = x + \sigma y, \qquad \eta = \gamma y, \tag{2.2}
$$

where

$$
\sigma = b/c, \quad \gamma = \sqrt{ac - b^2}/c.
$$

The positive definiteness of the strain energy which is

$$
ac - b^2 > 0,
$$

preserves the ellipticity of (2.1). An application of the affine transformation of (2.2) to (2.1) yields

$$
\begin{aligned} -\Delta\varphi^* &= f \quad \text{in} \quad R^*, \\ \varphi^* &= 0 \quad \text{on} \quad \partial R^*, \end{aligned} \tag{2.3}
$$

where $f = 2c/(ac - b^2)$ and Δ denotes the Laplacian operator. Here and in what follows superscripts * denotes the transformed equivalent isotropic boundary value problem. Our primary concern is to estimate the torsional rigidity of R^*, denoted by S^* which is defined in terms of the Dirichlet integral of φ^* (see [9] and [11])

$$S^* = \iint_{R^*} |\nabla \varphi^*|^2 d\xi d\eta,$$

where $\nabla \varphi^*$ is the gradient of φ^*. By applying Green's theorem, we can rewrite S^* as

$$S^* = \iint_{R^*} f\varphi^* d\xi d\eta.$$

Here we consider the well-known inequality for characterizing variational boundary value problem of (2.3),

$$\max_{u \in w_0^{1,2}} \frac{\left(\iint_{R^*} fud\xi d\eta\right)^2}{\iint_{R^*} |\nabla u|^2 d\xi d\eta} \leq S^* \leq \min_{-\Delta v = f} \iint_{R^*} |\nabla v|^2 d\xi d\eta, \tag{2.4}$$

where $w_0^{1,2}$ is an admissible Sobolev space (see [4], [8], [9] and [13]). By applying the first part of inequality (2.4), a suitable lower bound for S^* using the classical isoperimetric inequality, which is due to Payne and Weinberger [7] for the isotropic case, is given by

$$\begin{aligned} S^* &\geq \frac{(fA^*)^2}{8\pi} \left\{1 - 2\varphi^{*2}/(1 - \varphi^{*2}) - 4\varphi^{*4} \ln \varphi^*/(1 - \varphi^{*2})^2\right\} \\ &\equiv F(\varphi^{*2}), \end{aligned} \tag{2.5}$$

where

$$\varphi^{*2} = 1 - 4\pi A^*/L^{*2} \geq 0. \tag{2.6}$$

We note that this classical isoperimetric inequality (2.6) shows that the expression on the right of (2.6) is always nonnegative, and vanishes if and only if R^* is a circle. Here and below $(L^*,\ A^*)$ and (L, A) denote the perimeter and the area of the transformed and original cross-section respectively. In order to establish the inequality between L and L^*, we note

$$
\begin{aligned}
L^* &= \int_0^{2\pi} \sqrt{\xi'^2(t) + \eta'^2(t)}dt \\
&= \int_0^{2\pi} \sqrt{x'^2(t) + 2\sigma x(t)'y(t)' + (\sigma^2 + \gamma^2)y'^2(t)}dt \qquad (2.7)\\
&\le \int_0^{2\pi} \sqrt{(1 + \sigma/\epsilon)x'^2(t) + (\sigma^2 + \epsilon\sigma + \gamma^2)y'^2(t)}dt,
\end{aligned}
$$

with $0 \le t \le 2\pi$ and a prime means differentiation with respect to its argument t. Here we have used the weighted arithmetic-geometric mean inequality and ϵ is a positive optimal constant to be determined such that $1 + \sigma/\epsilon = \sigma^2 + \epsilon\sigma + \gamma^2$. Then we can write (2.7) as

$$L^* \le kL, \tag{2.8}$$

here

$$k = \left\{ \frac{a + c + (c^2 - 2ac + a^2 + 4b^2)^{1/2}}{2c} \right\}^{1/2}.$$

By virtue of (2.8) and $A^* = \gamma A$, we can rewrite (2.6) as

$$\varphi^*2 \le 1 - 4\pi\gamma A/(kL)^2 = \omega^2. \tag{2.9}$$

Since $F(\varphi^{*2})$ is a monotonic decreasing function of φ^{*2}, an insertion of (2.9) into (2.5) yields

$$S^* \ge \frac{A^2}{2\pi(ac - b^2)} \left\{ 1 - 2\omega^2/(1 - \omega^2) - 4\omega^4 \ln\omega/(1 - \omega^2)^2 \right\}, \tag{2.10}$$

which is the desired lower bound for the torsional rigidity of R. On the other hand, an upper bound for S^* in terms of A^* is estimated by the isoperimetric inequality $S^* \le f^2 A^{*2}/(8\pi) = f^2\gamma^2 A^2/(8\pi)$, which was proved by Pólya [8], and Pólya and Szegö [9] for the isotropic case. Then we have an upper bound $S \le A^2/\{2\pi(ac - b^2)\}$.

3. An upper bound by the complementary variational principles

In this section we would like to characterize the second part of inequality (2.4) by the complementary variational principles (see Stakgold [13]). Now we choose any function such that $-\Delta v = f$. First of all, we consider a particular solution to $-\Delta v = f$, that is, $\frac{-c}{2(ac-b^2)}(\xi^2 + \eta^2)$. Since we want to estimate an upper bound for the transformed torsional rigidity in terms of the second moments of the cross-section, we naturally choose a conjugate harmonic function in the form $\frac{c}{2(ac-b^2)}(\xi^2 - \eta^2)$ and $\frac{c}{ac-b^2}\xi\eta$. Then a trial function for computable constants k_1 and k_2 is of the form,

$$(3.1)\quad v = \frac{-c}{2(ac-b^2)}(\xi^2+\eta^2) + k_1\frac{c}{2(ac-b^2)}(\xi^2-\eta^2) + k_2\frac{c}{(ac-b^2)}\xi\eta.$$

Since this v clearly satisfies $-\Delta v = f$, selecting k_1 and k_2 to minimize $\iint_{R^*} |\nabla v|^2 d\xi d\eta$ we obtain

$$k_1 = \frac{I_\eta - I_\xi}{I_\xi + I_\eta}, \qquad k_2 = \frac{2I_{\xi\eta}}{I_\xi + I_\eta},$$

where

$$I_\eta = \iint_{R^*} \xi^2 d\xi d\eta, \quad I_\xi = \iint_{R^*} \eta^2 d\xi d\eta, \quad I_{\xi\eta} = \iint_{R^*} \xi\eta d\xi d\eta.$$

The optimal value of S^* is found to be

$$(3.2)\qquad S^* \le \frac{4c^2}{(ac-b^2)^2}\left(\frac{I_\xi I_\eta - I_{\xi\eta}^2}{I_\xi + I_\eta}\right).$$

To recover the original torsional rigidity S in R from the torsional rigidity S^* in R^*, noting the following identities

$$I_\eta = \gamma\left(I_y + 2\sigma_{xy} + \sigma^2 I_x\right), \quad I_\xi = \gamma^3 I_x, \quad I_{\xi\eta} = \gamma^2\left(I_{xy} + \sigma I_x\right),$$

we find the upper bound S in R

$$(3.3)\qquad S \le \frac{4}{\sqrt{ac-b^2}}\left(\frac{I_x I_y - I_{xy}^2}{aI_x + 2bI_{xy} + cI_y}\right),$$

where

$$I_y = \iint_R x^2 dxdy, \quad I_x = \iint_R y^2 dxdy, \quad I_{xy} = \iint_R xydxdy.$$

4. An elliptic inequality satisfying the maximum principle

We want to derive an upper bound by the maximum principle. We define $P = |\nabla\varphi^*|^2 + 2f\varphi^*$, and observe

$$\begin{aligned} P_{,k} &= 2\varphi^*_{,j}\varphi^*_{,jk} + 2f\varphi^*_{,k}, \\ \Delta P &= 2\varphi^*_{,jk}\varphi^*_{,jk} - 2f^2 \\ &\geq \frac{(P_{,k} - 2f\varphi^*_{,k})(P_{,k} - 2f\varphi^*_{,k})}{2|\nabla\varphi^*|^2} - 2f^2, \end{aligned}$$

where we have adopted the summation convention and employed Schwarz's inequality in the form

$$|\nabla\varphi^*|^2\varphi^*_{,jk}\varphi^*_{,jk} \geq \varphi^*_{,ki}\varphi^*_{,k}\varphi^*_{,ji}\varphi^*_{,j}.$$

This simplifies to

$$\Delta P + w_k P_{,k} \geq 0,$$

where w_k is singular at critical points of φ^*, i.e. at points where $\nabla\varphi^* = 0$. Thanks to this elliptic inequality, we can apply the maximum principle which is: either P attains its maximum on ∂R^* or at a critical point. In the former case $\frac{\partial P}{\partial n} \geq 0$ at the point on ∂R^* where P attains its maximum where n denotes the outward normal to ∂R^* (see for example Protter and Weinberger [10]). We want to assert that P attain its maximum at a critical point. To do this we can easily show that $\frac{\partial P}{\partial n} \leq 0$ on ∂R^* by Payne's argument [5]. According to Payne, for convex region we have either the maximum value of P cannot occur on ∂R^* if R^* is strictly convex or the maximum value of P must occur at the interior critical point where φ^* assumes its maximum value, if R^* is convex. In either case, then we have

$$P = |\nabla\varphi^*|^2 + 2f\varphi^* \leq 2f\varphi^*_M, \tag{4.1}$$

where φ^*_M is the maximum value of φ^* in R^*. Let M be the point where $\varphi^* = \varphi^*_M$ (There is only such point if R^* is convex (see Sperb [12]).) and Q a point on ∂R^* nearest to M. Let r measure the distance from M along the ray connecting M and Q. Certainly

$$-\frac{d\varphi^*}{dr} \leq |\nabla\varphi^*|$$

and therefore we rewrite (4.1) as

$$-\frac{d\varphi^*}{dr} \leq \sqrt{2f(\varphi_M^* - \varphi^*)},$$

or

$$\int_0^{\varphi_M^*} \frac{d\varphi^*}{\sqrt{2f(\varphi_M^* - \varphi^*)}} \leq \int_M^Q dr = \delta^*,$$

where δ^* is the distance from M to Q. Evaluating this integral we conclude

$$\varphi_M^* \leq \frac{f\delta^{*2}}{2} \leq \frac{fd^{*2}}{2}, \tag{4.2}$$

where d^* is the radius of the largest circle inscribed in R^*. Since $d^{*2} = \max(\xi^2 + \eta^2)$ for arbitrary point of origin in R^*, reverting to the original coordinate, we obtain

$$d^2 = \max\left\{(x + by/c)^2 + (ac - b^2)y^2/c^2\right\}.$$

Here $2d$ is the length of the major axis of the largest ellipse of the form

$$cx^2 + 2bxy + ay^2 = \text{constant},$$

which can be inscribed in R, with center at an arbitrary point in R. Let us observe also that integrating (4.1) over R^*, and using the definition of the torsional rigidity, we end up with

$$S^* + 2S^* \leq 2f\varphi_M^* A^*.$$

Then we obtain

$$S^* \leq \frac{f^2}{3} d^{*2} A^*,$$

as a result of (4.2). With the definition of f, d, and $A^* = \gamma A$, we arrive at the following upper estimate for the anisotropic torsional rigidity,

$$S \leq \frac{4c}{3(ac - b^2)^{3/2}} d^2 A.$$

In summary, employing the affine transformation (2.2) we obtain the sharp isoperimetric bounds for the torsional rigidity of the anisotropic beam and we can compile all the inequalities related to the isotropic torsion problem, which are also valid for the anisotropic case. As a check on our analysis, for the special isotropic case $a = c = 1$ and $b = 0$ the results of these isoperimetric inequalities recover those of the torsional rigidity of the isotropic beam (see [4] and [9]). We should remark that this analysis can be extended to the anisotropic torsion problem for multiply connected regions investigated by Payne [6].

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