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G/M/1 QUEUES WITH ERLANGIAN VACATIONS

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ABSTRACT. We consider a $G/M/1$ vacation model where the vacation time has k-stages generalized Erlang distribution. By using the methods of the shift operator and supplementary variable, we explicitly obtain the limiting probabilities of the queue length at arrival time points and arbitrary time points simultaneously. Operational calculus technique is used for solving non-homogeneous difference equations.

1. Introduction

In recent years there have been significant contributions to the theory of queue with server vacations. For complete reference on vacation models, see Doshi[8] and Takagi[19]. Vacation models have been widely used to model many problems in computer, communication and production system. Vacation models also are closely related to cyclic queues and retrial queues. $M/G/1$ vacation models under the various service disciplines were investigated by many authors ([4,7,9,11,13,15,16,17]). But $G/M/1$ type vacation models have received relatively little attention. Daniel and Krishnamoorthy[6] investigated $G/M/1$ vacation model under a limited service discipline by the matrix-geometric approach. Recently Tian, Zhang and Cao[20] investigated the $G/M/1$ vacation model under the exhaustive service discipline and derived queue length probabilities at arrival time points by the matrix-geometric approach and at arbitrary time points by the method of the semi-Markov process. Independently Choi and Park[1] obtained the same results for $G/M/1$ vacation model under the exhaustive service discipline by imbedded Markov chain approach. Under the exhaustive service disciplines there are two different vacation models which are called the multiple vacation model and the single vacation model (Doshi[8]). In the multiple vacation model, the

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server takes a vacation every time the system becomes empty or whenever the server returns from a vacation to find no waiting customers.

For $G/M/1$ type vacation model, it is always assumed ([1,6,20]) that vacation time has exponential distribution. In this paper we analyze the $G/M/1$ queue with multiple vacation discipline where the vacation time has k-stages generalized Erlang distribution. With the supplementary variable method, we explicitly obtain the queue length probabilities at arrival time points and arbitrary time points simultaneously. It is well known that shift operator $\mathcal{D}$ and its polynomial $f(\mathcal{D})$ are very useful tools for solving simple difference equations (see, for example, Gross and Harris[10]). In this paper we show that the operator method is still powerful technique even for complicated simultaneous non-homogeneous difference equations.

This paper is organized as follows. In the section 2, we discuss operational calculus about the shift operator. In section 3, queue length probabilities at arrival points and arbitrary points are derived explicitly by solving a non-homogeneous difference equation. In section 4, since our model for $k = 1$ is $G/M/1$ queue with exponential vacation, it is shown that our results by supplementary variable approach coincide with the corresponding known results obtained by imbedded Markov chain approach. (Choi and Park[1], Tien et al.[20]).

2. Operational calculus

In this section, we discuss operational calculus for later use. For a sequence $\{x_n\}$ of complex numbers, the right shift operator $\mathcal{D}$ is defined by $\mathcal{D}x_n = x_{n+1}$ for all n. If $f(z) = \alpha_0 + \alpha_1 z + \cdots + \alpha_k z^k$ is a polynomial with complex coefficients α_i, then the symbol $f(\mathcal{D}) = \alpha_0 + \alpha_1\mathcal{D} + \cdots + \alpha_k\mathcal{D}^k$ is naturally defined by

$$f(\mathcal{D}) \cdot x_n = \alpha_0 x_n + \alpha_1 x_{n+1} + \cdots + \alpha_k x_{n+k}.$$

It is well-known (see, for example, Gross and Harris[10], Spiegel[18]) that this type of polynomial $f(\mathcal{D})$ in $\mathcal{D}$ is used to find the general solution of difference equations.

Now we will introduce symbol $f(\mathcal{D})$ for other functions f in such a way that $f(\mathcal{D})$ has a natural meaning for particular geometric sequence

$\{\omega^n\}$. For $f(z) = \sum_{k=0}^{\infty} \alpha_k z^k$, it is natural to define $f(\mathcal{D}) = \sum_{k=0}^{\infty} \alpha_k \mathcal{D}^k$ by

$$f(\mathcal{D}) \cdot \omega^n = (\sum_{k=0}^{\infty} \alpha_k \mathcal{D}^k) \cdot \omega^n = f(\omega) \cdot \omega^n.$$

For instance, since $\exp(z) = \sum_{n=0}^{\infty} \frac{1}{k!} z^k$, it follows that

$$\exp(\mathcal{D}) \cdot \beta^n = \exp(\beta) \cdot \beta^n.$$

Let $a^*(\theta) = \int_0^{\infty} e^{-\theta x} a(x)\, dx$ be the Laplace transform of function $a(x)$. By the same reason as above, we have $a^*(\mathcal{D}) \cdot \beta^n = a^*(\beta) \cdot \beta^n$.

When $f(\mathcal{D}) \cdot x_n = \beta^n$, the inverse operator $\frac{1}{f(\mathcal{D})}$ of $f(\mathcal{D})$ is defined by $\frac{1}{f(\mathcal{D})} \cdot \beta^n = x_n$. For instance, since $(\mathcal{D} + \alpha) \cdot \beta^n = (\beta + \alpha) \cdot \beta^n$, it follows that

$$\left(\frac{1}{\mathcal{D} + \alpha}\right) \cdot \beta^n = \left(\frac{1}{\beta + \alpha}\right) \beta^n.$$

For a geometric sequence $\{\beta^n\}$, we summarize operator formula which are very useful in obtaining a particular solution of difference equations.

PROPOSITION 2.1. *For* $\alpha_1, \alpha_2, \beta \in \mathbf{C}$ *and* $m \in \mathbf{N}$,

(i) $(\alpha_1 \mathcal{D}^m + \alpha_2) \cdot \beta^n = (\alpha_1 \beta^m + \alpha_2) \cdot \beta^n$.

(ii) $\dfrac{1}{\alpha_1 \mathcal{D}^m + \alpha_2} \cdot \beta^n = \dfrac{1}{\alpha_1 \beta^m + \alpha_2} \cdot \beta^n$, *if* $\alpha_1 \beta^m + \alpha_2 \neq 0$.

(iii) $a^*(\alpha_1 + \alpha_2 \mathcal{D}^m) \cdot \beta^n = a^*(\alpha_1 + \alpha_2 \beta^m) \cdot \beta^n$.

(iv) $\dfrac{1}{\mathcal{D} - a^*(\alpha_1 + \alpha_2 \mathcal{D}^m)} \cdot \beta^n = \dfrac{1}{\beta - a^*(\alpha_1 + \alpha_2 \beta^m)} \cdot \beta^n$, *if* $\beta - a^*(\alpha_1 + \alpha_2 \beta^m) \neq 0$.

3. Queue length distribution

We consider a $G/M/1$ queueing system with server vacations. The interarrival times of calls are independently identical distributed with p.d.f. $a(x)$, mean $\frac{1}{\lambda}$ and Laplace transform $a^*(\theta)$. The service times are distributed exponentially with mean $\frac{1}{\mu}$. The vacation time has k-stages generalized Erlang distribution with parameters ν_i and $\nu_i \neq \nu_j$ for $i \neq j$. Since the statistical equilibrium conditions for both the queue with vacation and the queue without vacation are same [8], we always assume that $\rho = \frac{\lambda}{\mu} < 1$ in the remaining of this paper. We will investigate the distributions of the queue length in the system at arrival time points and at arbitrary time points simultaneously by the supplementary variable method. Here we take supplementary variable as the remaining interarrival time.

At an arbitrary time, the steady state of the system can be characterized by the following variables;

$$
\begin{aligned}
N &= \text{ the number of calls in the system;} \\
\tilde{A} &= \text{ the remaining interarrival time;} \\
\xi &= \begin{cases} 0, & \text{if server is busy;} \\ j, & \text{if server is in stage } j \text{ on vacation, } j = 1, 2, \cdots, k. \end{cases}
\end{aligned}
$$

Define

$$
\begin{aligned}
p_i(x)dx &= P(N = i, \xi = 0, \tilde{A} \in (x, x + dx]), \ i \geq 1, \\
q_{ij}(x)dx &= P(N = i, \xi = j, \tilde{A} \in (x, x + dx]), \ \ i \geq 0, \ 1 \leq j \leq k,
\end{aligned}
$$

and their Laplace transforms,

$$
\begin{aligned}
p_i^*(\theta) &= \int_0^\infty e^{-\theta x} p_i(x)dx, \\
q_{ij}^*(\theta) &= \int_0^\infty e^{-\theta x} q_{ij}(x)dx.
\end{aligned}
$$

Note that $q_{ij}^*(0)$ and $\frac{q_{ij}(0)}{\lambda}$, denoted by $q_{ij}^{(a)}$, are probabilities that there are i calls in the system and server is in stage j on vacation at arbitrary time points and arrival time points, respectively, and $p_i^*(0)$ and $\frac{p_i(0)}{\lambda}$, denoted by $p_i^{(a)}$, are probabilities that there are i calls in the system and server is busy at arbitrary time points and arrival time points, respectively.

By considering the steady state, we have the following system of differential difference equations;

(3.1a) $-p_1'(x) = -\mu p_1(x) + \mu p_2(x) + \nu_k q_{1k}(x),$

(3.1b) $-p_i'(x) = -\mu p_i(x) + \mu p_{i+1}(x) + \nu_k q_{ik}(x) + a(x)p_{i-1}(0),$
$i \geq 2,$

(3.1c) $-q_{01}'(x) = -\nu_1 q_{01}(x) + \mu p_1(x) + \nu_k q_{0k}(x),$

(3.1d) $-q_{i1}'(x) = -\nu_1 q_{i1}(x) + a(x)q_{i-11}(0), \quad i \geq 1,$

(3.1e) $-q_{0j}'(x) = -\nu_j q_{0j}(x) + \nu_{j-1} q_{0j-1}(x), \quad 2 \leq j \leq k,$

(3.1f) $-q_{ij}'(x) = -\nu_j q_{ij}(x) + \nu_{j-1} q_{ij-1}(x) + a(x)q_{i-1j}(0),$
$i \geq 1,\ 2 \leq j \leq k.$

By taking Laplace transform to the above equations, it follows that

(3.2a) $(\theta - \mu)p_1^*(\theta) + \mu p_2^*(\theta) = p_1(0) - \nu_k q_{1k}^*(\theta),$

(3.2b) $(\theta - \mu)p_i^*(\theta) + \mu p_{i+1}^*(\theta) = p_i(0) - a^*(\theta)p_{i-1}(0) - \nu_k q_{ik}^*(\theta),$
$i \geq 2,$

(3.2c) $(\theta - \nu_1)q_{01}^*(\theta) + \nu_k q_{0k}^*(\theta) = q_{01}(0) - \mu p_1^*(\theta),$

(3.2d) $(\theta - \nu_1)q_{i1}^*(\theta) = q_{i1}(0) - a^*(\theta)q_{i-11}(0), \quad i \geq 1,$

(3.2e) $(\theta - \nu_j)q_{0j}^*(\theta) + \nu_{j-1} q_{0j-1}^*(\theta) = q_{0j}(0), \quad 2 \leq j \leq k,$

(3.2f) $(\theta - \nu_j)q_{ij}^*(\theta) + \nu_{j-1} q_{ij-1}^*(\theta) = q_{ij}(0) - a^*(\theta)q_{i-1j}(0),$
$i \geq 1,\ 2 \leq j \leq k,$

By using the shift operator $\mathcal{D}$, the equations (3.2d) and (3.2f) can be written as

(3.3d) $(\theta - \nu_1)q_{i1}^*(\theta) = (\mathcal{D} - a^*(\theta))q_{i-11}(0), \quad i \geq 1,$

(3.3f) $(\theta - \nu_j)q_{ij}^*(\theta) + \nu_{j-1} q_{ij-1}^*(\theta) = (\mathcal{D} - a^*(\theta))q_{i-1j}(0),$
$i \geq 1,\ 2 \leq j \leq k.$

Letting $\theta = \nu_1$ in (3.3d), we have the following homogeneous equation

(3.4) $$(\mathcal{D} - \alpha_1)q_{i1}(0) = 0, \quad i \geq 0,$$

where $\alpha_1 = a^*(\nu_1)$.

The following lemmas are very often used for solving non-homogeneous difference equations.

LEMMA 3.1. ([10],p.307) *If $\frac{\lambda}{\mu} < 1$, then $z - a^*(\mu - \mu z) = 0$ has the unique real root γ between 0 and 1.*

LEMMA 3.2. *Let $\{x_n\}_{n=0}^{\infty}$ be an unknown sequence with $\sum_{n=0}^{\infty} |x_n| \leq 1$.*

(i) *A particular solution of difference equation $(\mathcal{D} - \delta) \cdot x_n = \beta^n$ with $\beta \neq \delta$ is given by*

$$x_n = \frac{1}{\beta - \delta} \cdot \beta^n.$$

(ii) *The general solution of homogeneous difference equation $(\mathcal{D} - \delta) \cdot x_n = 0$ with $|\delta| < 1$ is given by*

$$x_n = c\delta^n,$$

where c is arbitrary constant.

(iii) *A particular solution of difference equation $(\mathcal{D} - a^*(\mu - \mu\mathcal{D})) \cdot x_n = \beta^n$ with $\beta \neq \gamma$ is given by*

$$x_n = \frac{1}{\beta - a^*(\mu - \mu\beta)} \cdot \beta^n.$$

(iv) *If $\frac{\lambda}{\mu} < 1$, then the general solution of homogeneous difference equation $(\mathcal{D} - a^*(\mu - \mu\mathcal{D})) \cdot x_n = 0$ is given by*

$$x_n = c\gamma^n,$$

where c is arbitrary constant.

PROOF. (i) This is obtained by applying Proposition 2.1.(ii).
(ii) Clear.
(iii) This is obtained by applying Proposition 2.1.(iv).
(iv) In general, when γ_i is a root of $z - a^*(\mu - \mu z) = 0$, a solution of $(\mathcal{D} - a^*(\mu - \mu\mathcal{D})) \cdot x_n = 0$ is given by $x_n = c_i\gamma_i^n$. So the general solution is a linear combination of such solutions. But we require that a solution $\{x_n\}$ must satisfies $\sum_{n=0}^{\infty} |x_n| \leq 1$. To satisfy this condition, a root γ of $z - a^*(\mu - \mu z) = 0$ must be inside the unit circle. Since there is only one

root inside the unit circle of $z - a^*(\mu - \mu z) = 0$ under the assumption $\frac{\lambda}{\mu} < 1$, the general solution of $(\mathcal{D} - a^*(\mu - \mu\mathcal{D})) \cdot x_n = 0$ is given by $x_n = c\gamma^n$. □

By applying Lemma 3.2(ii), the solution of homogeneous equation (3.4) is given by

$$q_{i1}(0) = c_1\alpha_1^i, \quad i \geq 0, \tag{3.5}$$

where c_1 is arbitrary constant. Substitution of the equation (3.5) into (3.3d) yields

$$q_{i1}^*(\theta) = \frac{c_1}{\theta - \nu_1}(\alpha_1 - a^*(\theta))\alpha_1^{i-1}, \quad i \geq 1. \tag{3.6}$$

Next we will find $q_{i2}(0)$ and $q_{i2}^*(\theta)$. By inserting (3.6) into (3.3f), we obtain

$$\begin{aligned}(\theta - \nu_2)q_{i2}^*(\theta) =& (\mathcal{D} - a^*(\theta))q_{i-12}(0)\\ &- \frac{c_1\nu_1}{\theta - \nu_1}(\alpha_1 - a^*(\theta))\alpha_1^{i-1}, \quad i \geq 1.\end{aligned} \tag{3.7}$$

Letting $\theta = \nu_2$ in (3.7), we have

$$(\mathcal{D} - \alpha_2)q_{i2}(0) = \frac{c_1\nu_1}{\nu_2 - \nu_1}(\alpha_1 - \alpha_2)\alpha_1^i, \quad i \geq 0, \tag{3.8}$$

where $\alpha_2 = a^*(\nu_2)$. By applying Lemma 3.2(i), a particular solution of (3.8) is given by

$$q_{i2}^{(p)}(0) = \frac{c_1}{\nu_2 - \nu_1}\nu_1\alpha_1^i. \tag{3.9}$$

By Lemma 3.2(iv), the general solution of homogeneous difference equation of (3.8) is given by

$$q_{i2}^{(h)}(0) = c_2\alpha_2^i, \tag{3.10}$$

where c_2 is arbitrary constant. Since the general solution of non-homogeneous difference equation (3.8) is the sum of the general solution of homogeneous equation and particular solution, the general solution of (3.8) is given by

$$q_{i2}(0) = c_2\alpha_2^i + \frac{c_1}{\nu_2 - \nu_1}\nu_1\alpha_1^i, \quad i \geq 0. \tag{3.11}$$

By substituting (3.11) into (3.7), we have for $i \geq 1$

$$\begin{aligned} q_{i2}^*(\theta) =& \frac{c_2}{\theta - \nu_2}(\alpha_2 - a^*(\theta))\alpha_2^{i-1} \\ &+ \frac{c_1}{(\theta - \nu_1)(\nu_2 - \nu_1)}\nu_1(\alpha_1 - a^*(\theta))\alpha_1^{i-1}, \quad i \geq 1. \end{aligned} \tag{3.12}$$

By following the same recursive way that we have obtained $q_{i2}(0)$ and $q_{i2}^*(\theta)$ from $q_{i1}(0)$ and $q_{i1}^*(\theta)$, we can obtain $q_{ij}(0)$ and $q_{ij}^*(\theta)$ as follows;

$$q_{ij}(0) = \sum_{n=1}^{j} c_n \prod_{l=n+1}^{j} \frac{\nu_{l-1}}{\nu_l - \nu_n}\alpha_n^i, \quad i \geq 0, \ 1 \leq j \leq k, \tag{3.13}$$

$$q_{ij}^*(\theta) = \sum_{n=1}^{j} c_n(\alpha_n - a^*(\theta))\frac{1}{\theta - \nu_n}\prod_{l=n+1}^{j}\frac{\nu_{l-1}}{\nu_l - \nu_n}\alpha_n^{i-1}, \quad i \geq 1, \ 1 \leq j \leq k, \tag{3.14}$$

where c_i are arbitrary constants and with convention $\prod_{l=j+1}^{j}\frac{\nu_{l-1}}{\nu_l - \nu_n} = 1$.

Now we determine $p_i(0)$ and $p_i^*(\theta)$. By using the shift operator $\mathcal{D}$, the equation (3.2b) can be written as

$$(\theta - \mu + \mu\mathcal{D})p_i^*(\theta) = (\mathcal{D} - a^*(\theta))p_{i-1}(0) - \nu_k q_{ik}^*(\theta), \quad i \geq 1. \tag{3.15}$$

By considering the imbedded Markov chain at arrival time points, we obtain the following relation

$$\begin{aligned} p_i(0) =& \sum_{j=i-1}^{\infty} p_j(0)b_{j-i+1} + \sum_{j=i-1}^{\infty}\sum_{l=1}^{k} q_{jl}(0) \\ &\cdot \int_0^{\infty}\int_0^{t} a(t)\frac{(\mu(t-x))^{j-i+1}}{(j-i+1)!}e^{-\mu(t-x)}dF_l(x)dt, \end{aligned} \tag{3.16}$$

where $F_l(x)$ is generalized Erlang distribution with parameter ν_l, ν_{l+1}, $\cdots$, ν_k, and $b_n = \int_0^\infty \frac{e^{-\mu t}(\mu t)^n}{n!} a(t)dt$: probability of n service completions during an interarrival time. By the induction on l, we see that (see, Appendix A)

$$\int_0^\infty \int_0^t a(t)e^{-\mu(1-\alpha_i)(t-x)} dF_l(x)dt$$

$$(3.17) \qquad = \prod_{h=l}^{k} \nu_h \Big[\prod_{m=l}^{k} \frac{1}{\nu_m - \mu(1-\alpha_i)} a^*(\mu - \mu\alpha_i) - \sum_{m=l}^{k} \frac{\alpha_m}{\nu_m - \mu(1-\alpha_i)} \prod_{\substack{n=l \\ n\neq m}}^{k} \frac{1}{\nu_n - \nu_m} \Big].$$

Note that when we apply the induction, we have used well-known property of convolution(Chung[3], p.146) and identity

$$1 = \sum_{m=l-1}^{k} \prod_{\substack{n=l-1 \\ n\neq m}}^{k} \frac{\nu_n - \mu_i}{\nu_n - \nu_m},$$

$\mu_i = \mu(1-\alpha_i)$ (Conte and Boor[5], p.38). Also we easily see that

$$\sum_{n=0}^{\infty} b_n z^n = a^*(\mu - \mu z)$$

and hence

$$\sum_{j=i-1}^{\infty} p_j(0) b_{j-i+1} = (\sum_{n=0}^{\infty} b_n \mathcal{D}^n) p_{i-1}(0) = a^*(\mu - \mu\mathcal{D}) p_{i-1}(0).$$

By substituting (3.17) into (3.16) and using above fact, we obtain (see, Appendix B)

$$(3.18) \qquad (\mathcal{D} - a^*(\mu - \mu\mathcal{D})) p_i(0) = \nu_k \sum_{n=1}^{k} c_n(\alpha_n - a^*(\mu - \mu\alpha_n)) \cdot \frac{1}{\mu(1-\alpha_n) - \nu_n} \prod_{l=n+1}^{k} \frac{\nu_{l-1}}{\nu_l - \nu_n} \alpha_n^i, \quad i \geq 1.$$

Note that the another formal way to obtain (3.18) is to substitute $\theta = \mu - \mu\mathcal{D}$ into (3.15). By Lemma 3.2(iii), a particular solution of (3.18) is given by

$$(3.19) \qquad p_i^{(p)}(0) = \nu_k \sum_{n=1}^{k} \frac{c_n}{\mu(1-\alpha_n)-\nu_n} \prod_{l=n+1}^{k} \frac{\nu_{l-1}}{\nu_l - \nu_n} \alpha_n^i.$$

By Lemma 3.1 and Lemma 3.2(iv), the general solution of homogeneous difference equation of (3.18) is given by

$$(3.20) \qquad p_i^{(h)}(0) = c_{k+1}\gamma^i,$$

where c_{k+1} is arbitrary constant. Thus the general solution of (3.18) is given by

$$(3.21) \quad p_i(0) = c_{k+1}\gamma^i + \nu_k \sum_{n=1}^{k} \frac{c_n}{\mu(1-\alpha_n)-\nu_n} \prod_{l=n+1}^{k} \frac{\nu_{l-1}}{\nu_l - \nu_n} \alpha_n^i, \quad i \geq 1.$$

Next we find out $p_i^*(\theta)$. Since $\frac{\mu-\theta}{\mu}$ is the unique zero of $\theta - \mu + \mu z = 0$ for fixed θ with $Re(\theta) \geq 0$, the general solution of homogeneous difference equation of (3.15) is $d\left(\frac{\mu-\theta}{\mu}\right)^i$ where d is arbitrary constant. A particular solution of (3.15) is obtained by inserting (3.14) and (3.21) into (3.15) and applying Lemma 3.2(i). Thus the general solution of (3.15) is

$$\begin{aligned} p_i^*(\theta) =& d\left(\frac{\mu-\theta}{\mu}\right)^i + \frac{c_{k+1}}{\theta-\mu+\mu\gamma}(\gamma - a^*(\theta))\gamma^{i-1} \\ (3.22) \qquad & + \nu_k \sum_{n=1}^{k} c_n(\alpha_n - a^*(\theta)) \frac{1}{(\theta-\nu_n)(\mu(1-\alpha_n)-\nu_n)} \\ & \cdot \prod_{l=n+1}^{k} \frac{\nu_{l-1}}{\nu_l - \nu_n} \alpha_n^{i-1}, \quad i \geq 2. \end{aligned}$$

But when $\theta = 0$ in (3.22), the first term of right hand side of (3.22) is d.

Since $\sum_{i=1}^{\infty} p_i^*(0) \leq 1$, d must be 0. Hence we have

$$\tag{3.23} \begin{aligned} p_i^*(\theta) =& \frac{c_{k+1}}{\theta-\mu+\mu\gamma}(\gamma - a^*(\theta))\gamma^{i-1} + \nu_k \sum_{n=1}^{k} c_n(\alpha_n - a^*(\theta)) \\ & \cdot \frac{1}{(\theta-\nu_n)(\mu(1-\alpha_n)-\nu_n)} \prod_{l=n+1}^{k} \frac{\nu_{l-1}}{\nu_l-\nu_n}\alpha_n^{i-1}, \ i \geq 2. \end{aligned}$$

Finally we determine the constants c_i, $i = 1, 2, \cdots, k+1$. For this purpose, we will find $k+1$ equations involving c_i, among which k equations come from boundary constants (3.2a), (3.2c) and (3.2e) and other equation comes from the following relation.

$$\tag{3.24} \lambda = \sum_{i=1}^{\infty} p_i(0) + \sum_{i=0}^{\infty}\sum_{j=1}^{k} q_{ij}(0).$$

Thus from (3.24) we obtain one equation;

$$\tag{3.25} \lambda = c_{k+1}\frac{\gamma}{1-\gamma} + \nu_k \sum_{n=1}^{k} \frac{c_n(\mu-\nu_n)}{\nu_n(\mu(1-\alpha_n)-\nu_n)} \prod_{l=n+1}^{k} \frac{\nu_{l-1}}{\nu_l-\nu_n}.$$

Letting $\theta = \mu$ in (3.2a) yields

$$p_1(0) = \nu_k q_{1k}^*(\mu) - \mu p_2^*(\mu).$$

By substituting (3.14), (3.21) and (3.22) into above equation, we have another equation;

$$\tag{3.26} c_{k+1} + \nu_k \sum_{n=1}^{k} \frac{c_n}{\mu(1-\alpha_n)-\nu_n} \prod_{l=n+1}^{k} \frac{\nu_{l-1}}{\nu_l-\nu_n} = 0.$$

To find the remaining $k-1$ equations, we need the explicit expression of $p_1^*(\theta)$, which can be obtained from (3.2a) as follows;

$$\tag{3.27} \begin{aligned} p_1^*(\theta) = \frac{1}{\theta-\mu} \Bigg\{ & c_{k+1}\frac{\theta-\mu+\mu a^*(\theta)}{\theta-\mu+\mu\gamma}\gamma \\ & + \nu_k \sum_{n=1}^{k} c_n \frac{(\theta-\mu)\alpha_n + (\mu-\nu_n)a^*(\theta)}{(\theta-\nu_n)(\mu(1-\alpha_n)-\nu_n)} \prod_{l=n+1}^{k} \frac{\nu_{l-1}}{\nu_l-\nu_n} \Bigg\}. \end{aligned}$$

If we eliminate $q^*_{01}(\theta)$ from (3.2c) and (3.2e) with $j = 2$, then we have

$$(3.28) \quad \begin{aligned} &(\theta-\nu_1)(\theta-\nu_2)q^*_{02}(\theta) - \nu_1\nu_k q^*_{0k}(\theta) \\ &= (\theta-\nu_1)q_{02}(0) - \nu_1(q_{01}(0) - \mu p^*_1(\theta)). \end{aligned}$$

Next we eliminate $q^*_{02}(\theta)$ from (3.28) and (3.2e) with $j = 3$. By keeping on this procedure, we have

$$(3.29) \quad Z(\theta)q^*_{0k}(\theta) = \Phi(\theta),$$

where

$$\begin{aligned} Z(\theta) &= \prod_{i=1}^{k}(\theta-\nu_i) - (-1)^k \prod_{i=1}^{k}\nu_i, \\ \Phi(\theta) &= \sum_{n=0}^{k-2}(-1)^n \prod_{i=1}^{k-n-1}(\theta-\nu_i) \prod_{j=k-n}^{k-1}\nu_j\, q_{0k-n}(0) \\ &\quad - (-1)^k \prod_{i=1}^{k-1}\nu_i(c_1 - \mu p^*_1(\theta)). \end{aligned}$$

$Z(\theta)$ is clearly a polynomial of degree k whose roots shall be denoted by $\theta_1, \theta_2, \cdots, \theta_r$ with multiplicity $n_1, n_2, \cdots, n_r$, respectively, with $n_1 + \cdots + n_r = k$. It is directly seen that 0, called it θ_1, is a zero of $Z(\theta)$ and its multiplicity is one. Furthermore it can be shown from the form of the polynomial and Rouche's theorem that real parts of $\theta_2, \cdots, \theta_k$ are positive (for the similar argument, see, for example, Kleinrock[14], p. 293). Since $q^*_{0k}(\theta)$ is analytic function for $Re(\theta) > 0$, $\Phi(\theta)$ must satisfy

$$(3.30) \quad \frac{\partial^{k_i}}{\partial\theta^{k_i}}\Phi(\theta)|_{\theta=\theta_i} = 0, \quad \text{for } 1 \le i \le r, \quad 0 \le k_i \le n_i - 1,$$

which are $k-1$ equations. Note that $\Phi(0) = 0$ is identity equation for c_i. Thus we have obtained $k+1$ different equations (3.25), (3.26) and (3.30) with $k+1$ unknown numbers c_i, $i = 1, \cdots, k+1$, from which c_i ($i = 1, 2, \cdots, k+1$) can be found. Thus we have obtained the following;

THEOREM 3.3. (i) *The steady state probabilities* $q_{ij}^{(a)}$ *and* $p_i^{(a)}$ *that an arrival sees* i *calls in the system and the server is in stage* j *on vacation and busy respectively are given by*

$$q_{ij}^{(a)} = \frac{1}{\lambda}\left\{\sum_{n=1}^{j} c_n \prod_{l=n+1}^{j} \frac{\nu_{l-1}}{\nu_l - \nu_n}\alpha_n^i\right\},\ i \geq 0,\ 1 \leq j \leq k,$$

$$p_i^{(a)} = \frac{1}{\lambda}\left\{c_{n+1}\gamma^i + \nu_k \sum_{n=1}^{k} \frac{c_n}{\mu(1-\alpha_n)-\nu_n} \prod_{l=n+1}^{k} \frac{\nu_{l-1}}{\nu_l - \nu_n}\alpha_n^i\right\},\ i \geq 1.$$

(ii) *The steady state probabilities* $q_{ij}^*(0)$ *and* $p_i^*(0)$ *that there are* i *calls in the system and the server is in stage* j *on vacation and busy at arbitrary time points respectively are given by*

$$q_{ij}^*(0) = \sum_{n=1}^{j} c_n(1-\alpha_n)\frac{1}{\nu_n} \prod_{l=n+1}^{j} \frac{\nu_{l-1}}{\nu_l - \nu_n}\alpha_n^{i-1},\ i \geq 1,\ 1 \leq j \leq k,$$

$$p_i^*(0) = \frac{c_{k+1}}{\mu - \mu\gamma}(1-\gamma)\gamma^{i-1} + \nu_k \sum_{n=1}^{k} c_n(1-\alpha_n)\frac{\prod_{l=n+1}^{k}\frac{\nu_{l-1}}{\nu_l-\nu_n}\alpha_n^{i-1}}{\nu_n(\mu(1-\alpha_n)-\nu_n)},\ i \geq 2,$$

$$p_1^*(0) = \frac{\nu_k}{\mu}\sum_{n=1}^{k}\frac{c_n}{\nu_n}\prod_{l=n+1}^{k}\frac{\nu_{l-1}}{\nu_l - \nu_n},$$

and $q_{0j}^*(0)$ *are obtained from (3.2c) and (3.2e).*

4. Special case

Since our model for $k = 1$ is $G/M/1$ queue with exponential vacation, we will show that theorem 3.3 match with known result for $G/M/1$ queue with exponential vacation (Choi and Park[1], Tian et al.[20]). Now we determine the constants c_1 and c_2. From (3.25) and (3.26), we obtain 2 equations involving c_1 and c_2

$$\frac{c_1(\mu - \nu_1)}{\mu(1-\alpha_1)-\nu_1} + \frac{c_2\gamma}{1-\gamma} = \lambda, \tag{4.1a}$$

$$\frac{c_1\nu_1}{\mu(1-\alpha_1)-\nu_1} + c_2 = 0. \tag{4.1b}$$

Solving the above simultaneous equations yields

$$c_1 = \lambda(1-\gamma)\sigma, \tag{4.2a}$$

$$c_2 = \lambda(1-\gamma)\sigma\beta, \tag{4.2b}$$

where $\sigma = \frac{\mu(1-\alpha_1)-\nu_1}{\mu(1-\gamma)-\nu_1}$ and $\beta = \frac{-\nu_1}{\mu(1-\alpha_1)-\nu_1}$. Then by Theorem 3.3(i),

$$p_i^{(a)} = (1-\gamma)\sigma\beta(\gamma^i - \alpha_1^i), \quad i \geq 1, \tag{4.3a}$$

$$q_{i1}^{(a)} = (1-\gamma)\sigma\alpha_1^i, \quad i \geq 0, \tag{4.3b}$$

and by Theorem 3.3(ii),

$$p_i^*(0) = \lambda(1-\gamma)\sigma\beta\left[\frac{\gamma^{i-1}}{\mu} - \frac{(1-\alpha_1)\alpha_1^{i-1}}{\nu_1}\right], \quad i \geq 2, \tag{4.4a}$$

$$p_1^*(0) = \frac{\lambda}{\mu}(1-\gamma)\sigma, \tag{4.4b}$$

$$q_{i1}^*(0) = \frac{\lambda}{\nu_1}(1-\alpha_1)(1-\gamma)\sigma\alpha_1^{i-1}, \quad i \geq 1. \tag{4.4c}$$

The above result agrees with the one for $G/M/1$ with exponential vacation (Choi and Park[1], Tian et al.[20]).

Appendix A. Proof of (3.17)

(3.17) will be proved by induction on l. (3.17) is true for $l = k$, since

$$\begin{aligned}
&\int_0^t e^{-s(t-x)} dF_k(x)dt \\
&= \int_0^t \nu_k e^{-\nu_k x} e^{-s(t-x)} dx \\
&= \frac{\nu_k}{\nu_k - s} e^{-st} - \frac{\nu_k}{\nu_k - s} e^{-\nu_k t} \\
&= \prod_{h=k}^{k} \nu_h \left[\prod_{m=k}^{k} \frac{e^{-st}}{\nu_m - s} - \sum_{m=k}^{k} \frac{e^{-\nu_m t}}{\nu_m - s} \prod_{\substack{n=k \\ n \neq m}}^{k} \frac{1}{\nu_n - \nu_m} \right].
\end{aligned}$$

Assume that (3.17) holds for $l = j$, *i.e.*,

$$\int_0^t e^{-s(t-x)} dF_j(x) = \prod_{h=j}^{k} \nu_h \left[\prod_{m=j}^{k} \frac{e^{-st}}{\nu_m - s} - \sum_{m=j}^{k} \frac{e^{-\nu_m t}}{\nu_m - s} \prod_{\substack{n=j \\ n \neq m}}^{k} \frac{1}{\nu_n - \nu_m} \right].$$

Now we will show that (3.17) is true for $l = j - 1$. By the property of convolution (Chung[3], p.146), we can write

$$\int_0^t e^{-s(t-x)} dF_{j-1}(x) = \int_0^t \nu_{j-1} e^{-\nu_{j-1} x} \int_0^{t-x} e^{-s(t-x-y)} dF_j(y)\, dx.$$

Now by induction hypothesis

$$\int_0^t e^{-s(t-x)} dF_{j-1}(x) = \prod_{h=j-1}^{k} \nu_h \left[\prod_{m=j-1}^{k} \frac{e^{-st} - e^{-\nu_{j-1} t}}{\nu_m - s} \right.$$
$$\left. - \sum_{m=j}^{k} \frac{e^{-\nu_m t} - e^{-\nu_{j-1} t}}{\nu_m - s} \prod_{\substack{n=j-1 \\ n \neq m}}^{k} \frac{1}{\nu_n - \nu_m} \right].$$

Thus to show that (3.17) holds for $l = j - 1$, it is enough to show that

$$\prod_{m=j-1}^{k} \frac{1}{\nu_m - s} - \sum_{m=j}^{k} \frac{1}{\nu_m - s} \prod_{\substack{n=j-1 \\ n \neq m}}^{k} \frac{1}{\nu_n - \nu_m} = \frac{1}{\nu_{j-1} - s} \prod_{n=j}^{k} \frac{1}{\nu_n - \nu_{j-1}},$$

which is equivalent to

$$\prod_{m=j-1}^{k} \frac{1}{\nu_m - s} = \sum_{m=j-1}^{k} \frac{1}{\nu_m - s} \prod_{\substack{n=j-1 \\ n \neq m}}^{k} \frac{1}{\nu_n - \nu_m},$$

i.e.,

$$\sum_{m=j-1}^{k} \prod_{\substack{n=j-1 \\ n \neq m}}^{k} \frac{\nu_n - s}{\nu_n - \nu_m} = 1.$$

By Lagrange formula for the interpolating polynomial (see Conte and Boor[5], p. 38), the above equation always holds.

Appendix B. Proof of (3.18)

By using (3.17), we obtain

$$\sum_{l=1}^{k}\sum_{j=n-1}^{\infty} q_{jl}(0)\int_0^{\infty}\int_0^{t} a(t)\frac{(\mu_i(t-x))^{j-n+1}}{(j-n+1)!}e^{-\mu_i(t-x)}dF_l(x)dt$$
$$=\sum_{i=1}^{k}\sum_{l=i}^{k} c_i\alpha_i^{n-1}\prod_{j=i}^{k}\nu_j\prod_{h=i+1}^{l}\frac{1}{\nu_h-\nu_i}$$
$$\cdot\left[\prod_{m=l}^{k}\frac{a^*(\mu_i)}{\nu_m-\mu_i}-\sum_{m=l}^{k}\frac{\alpha_m}{\nu_m-\mu_i}\prod_{\substack{p=l\\p\neq m}}^{k}\frac{1}{\nu_p-\nu_m}\right],$$

where $\mu_i=\mu(1-\alpha_i)$. To show that (3.18) holds, it is enough to show that

$$\text{(i)}\ \sum_{l=i}^{k}\prod_{h=i+1}^{l}\frac{1}{\nu_h-\nu_i}\prod_{m=l}^{k}\frac{1}{\nu_m-\mu_i}=\frac{1}{\nu_i-\mu_i}\prod_{l=i+1}^{k}\frac{1}{\nu_l-\nu_i},$$

$$\text{(ii)}\ \sum_{l=i+1}^{k}\sum_{m=l}^{k}\frac{\alpha_m}{\nu_m-\mu_i}\prod_{h=i+1}^{l}\frac{1}{\nu_l-\nu_i}\prod_{\substack{p=l\\p\neq m}}^{k}\frac{1}{\nu_p-\nu_m}$$
$$+\sum_{m=i+1}^{k}\frac{\alpha_m}{\nu_m-\mu_i}\prod_{\substack{p=l\\p\neq m}}^{k}\frac{1}{\nu_p-\nu_m}=0.$$

For (i) and (ii), it is enough to show that the following (i)′ and (ii)′ hold respectively

$$\text{(i)}'\ \sum_{i=1}^{m}\prod_{j=2}^{i}\frac{1}{\nu_j-\nu_1}\prod_{j=i}^{m}\frac{1}{\nu_j-\mu_1}=\frac{1}{\nu_1-\mu_1}\prod_{i=2}^{m}\frac{1}{\nu_i-\nu_1},$$

$$\text{(ii)}'\ \sum_{i=2}^{m}\prod_{j=2}^{i-1}\frac{\nu_j-\nu_m}{\nu_j-\nu_1}\cdot\frac{\nu_m-\nu_1}{\nu_i-\nu_1}=1.$$

The above equations (i)′ and (ii)′ are proved by the induction.

References

1. Choi, B. D. and Park, K. K., *G/M/1 vacation model with exhaustive service*, Communications of Kor. Math. Soc. **6** (1991), 267-281.
2. Choi, B. D. and Park, K. K., *The M/G/1 retrial queue with Bernoulli schedule*, Queueing Systems **7** (1990), 219-227.
3. Chung, K. L., *A course in probability theory*, Academic Press., 1968.
4. Cramer, M., *Stationary distributions in a queueing system with vacation times and limited service*, Queueing Systems **4** (1989), 57-68.
5. Conte, S. D. and Boor, C. D., *Elementary numerical analysis*, Mcgraw Hill, 1981.
6. Daniel, J. T. and Krishnamoorhty, A., *G/M/1 queue with rest periods*, Optimization **17** (1986), 535-543.
7. Doshi, B. T., *A note on stostastic decomposition in a G/G/1 queue with vacation or set up times*, J. Appl. Prob. **22** (1985), 419-428.
8. Doshi, B. T., *Queueing systems with vacation-a servey*, Queueing Systems **1** (1986), 29-66.
9. Fuhrmann, S.W. and Cooper, R.B., *Stochastic decomposition in an M/G/1 queue with generalized vacations*, Oper. Res. **33** (1985), 1117-1129.
10. Gross, D. and Harris, C.M., *Fundamental of Queueing Theory*, John Wiley & Sons, 1985.
11. Heyman, D.P., *Optimal operating policies for M/G/1 queueing system*, Oper. Res. **16** (1980), 362-382.
12. Hokstad, B.T., *A supplementary variable technique applied to the M/G/1 queue*, Scand. J. Stat. **2** (1986), 95-103.
13. Keilson, J. and Servi, L.D., *Oscillating random walk models for GI/G/1 vacation systems with Bernoulli schedules*, J. Appl. Prob. **23** (1986), 790-802.
14. Kleinrock, L., *Queueing Systems I*, John Wiley & Sons, 1975.
15. Lee, T. T., *M/G/1/N queue with vacation time and exhaustive service discipline*, Oper. Res. **32** (1984), 774-784.
16. Levy, Y. and Yechalli, U., *Utilization of idle time in an M/G/1 queueing system*, Management Sci. **22** (1975), 202-211.
17. Ramaswamy, R. and Servi, L. D., *The busy period of the M/G/1 vacation model with a Bernoulli schedules*, Comm. Statist. Stochastic Model **4** (1988), 507–521.
18. Spiegel, M. R., *Schaum's Outline of Theory and Problems of Calculus of Finite Differences and Difference quations*, Mcgraw Hill, 1971.
19. Takagi, H., *Queueing Analysis: A Foundation of Performance Evaluation, Vol 1: Vacation and Priority Systems, Part 1*, North-Holland, 1991.
20. Tian, N. Zhang, D. and Cao, C., *The GI/M/1 queue with exponential vacations*, Queueing Systems **5** (1989), 331-344.

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