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A NOTE ON SUMS OF RANDOM VECTORS WITH VALUES IN A BANACH SPACE

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ABSTRACT. Let $\{X_n : n = 1, 2, \cdots\}$ be a sequence of pairwise independent identically distributed random vectors taking values in a separable Hilbert space H such that $E\|X_1\| = \infty$. Let $S_n = X_1 + X_2 + \cdots + X_n$ and for any real α with $0 < \alpha < 1$ define a sequence $\{\gamma_n(\alpha)\}$ as $\gamma_n(\alpha) = \inf\{r : P(\|S_n\| \leq r) \geq \alpha\}$. Then

$$\limsup_{n \to \infty} \|S_n\|/\gamma_n(\alpha) = \infty$$

holds. This is a generalization of Vvedenskaya[2].

We consider a sequence of random vectors $\{X_n : n = 1, 2, \cdots\}$, taking values in a separable Banach space B. We assume that these vectors are nonzero with positive probabilities. That is, $P\{\|X_n\| > 0\} > 0$.
Let $S_n = X_1 + X_2 + \cdots + X_n$ and for any real α with $0 < \alpha < 1$ define a sequence $\{\gamma_n(\alpha)\}$ as

$$\gamma_n(\alpha) = \inf\{r : P(\|S_n\| \leq r) \geq \alpha\}.$$

It follows from well-known results on decrease of the concentration that if $\{X_n : n = 1, 2, \cdots\}$ satisfies some conditions, (for example, $\{X_n : n = 1, 2, \cdots\}$ is independent (pairwise independent) identically distributed), then $P(\|S_n\| = 0) \to 0$ as $n \to \infty$, and thus for any α the inequality $\gamma_n(\alpha) > 0$ holds beginning with some n. Under the set-up Vvedenskaya[2] posed the problem whether

$$\limsup_{n \to \infty} \|S_n\|/\gamma_n(\alpha) = \infty \tag{1}$$

and proved the following:

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THEOREM. (Vvedenskaya [2]) *Let* $\{X_n : n = 1, 2, \cdots\}$ *be a sequence of independent symmetrically and identically distributed random vectors taking values in a separable Hilbert space H such that* $E\|X_1\| = \infty$. *Then for any* α *with* $0 < \alpha < 1$, *(1) holds*

In this note, the same problems will be considered and proved under the much weaker condition than Vvedenskaya's. Throughout the paper, we assume that $\{X_n : n = 1, 2, \cdots\}$ is a sequence of ***pairwise independent identically distributed random vectors taking values in a separable Banach space B***. We begin with the following lemma whose proof, using Theorem 4.2.5 [1], is similiar to that of Lemma 1 [2]

LEMMA 1. *Let* $\{X_n : n = 1, 2, \cdots\}$ *be a sequence of pairwise independent identically distributed random vectors taking values in a separable Banach space B and let* $\{b_n : n = 1, 2, \cdots\}$ *a sequence of positive numbers such that* $\{b_n/n\}$ *is nondecreasing. Then*

(1) $\sum_{n=1}^{\infty} P(\|X_1\| > b_n) = \infty$ *implies* $\limsup_{n\to\infty} \|X_n\|/b_n = \infty$ *a.s. and*

(2) $\sum_{n=1}^{\infty} P(\|X_1\| > b_n) < \infty$ *implies* $\lim_{n\to\infty} \|X_n\|/b_n = 0$ *a.s.*.

Next lemma is crucial to the proof of the main theorem whose proof is quite elementary, compared to that of Vvedenskaya's.

LEMMA 2. *Let* $\{X_n : n = 1, 2, \cdots\}$ *and* $\{b_n : n = 1, 2, \cdots\}$ *be as in Lemma 1, and let* $E\|X_1\| = \infty$. *Then*

(1) $\sum_{n=1}^{\infty} P(\|X_1\| > b_n) = \infty$ *implies* $\limsup_{n\to\infty} \|S_n\|/b_n = \infty$ *a.s.*

(2) $\sum_{n=1}^{\infty} P(\|X_1\| > b_n) < \infty$ *implies for some further subsequence* $\{b_{n_k}\}$ *of any subsequence of* $\{b_n\}$, $\lim_{n\to\infty} \|S_{n_k}\|/b_{n_k} = 0$ *a.s.*.

PROOF. Since

$$\|X_n\|/b_n = \|S_n - S_{n-1}\|/b_n \le \|S_n\|/b_n + \|S_{n-1}\|/b_n,$$

(1) of Lemma 1 implies (1). Assume that

$$\sum_{n=1}^{\infty} P(\|X_1\| > b_n) < \infty.$$

Let

$$X'_k = \begin{cases} X_k & \text{if} \quad \|X_n\| \le b_n \\ 0 & \text{if} \quad \|X_k\| > b_k. \end{cases}$$

Then by Borel-Cantelli lemma, we conclude that

$$\|S_n\| = \|\sum_{k=1}^{n} X'_k\| + O(1) \qquad a.s.$$

To prove (2) it suffices to show that

$$\lim_{n\to\infty} \sum_{k=1}^{n} E\|X'_k\|/b_n = 0.$$

(see [1] ex.7. p75). Now

$$b_n^{-1} \sum_{k=1}^{n} E\|X'_k\| = b_n^{-1} \sum_{k=1}^{n} \int_{|x|\le b_k} |x| dF(x)$$

where $F(x) = P(\|X_1\| \le x)$. Clearly for any $N < n$, it is bounded by

$$\text{(2)} \qquad n b_n^{-1}(b_N + \int_{b_N<|x|<b_n} |x| dF(x)).$$

Since

$$E\|X_1\| = \infty \qquad \text{and} \qquad \sum_{n=1}^{\infty} P(\|X_1\| > b_n) < \infty,$$

$b_n n^{-1}$ cannot be bounded. Hence for fixed N the term $(n/b_n)b_N$ in (2) tends to 0 as $n \to \infty$ and the rest is bounded by

$$n b_n^{-1} \sum_{j=N+1}^{n} b_j \int_{b_{j-1}\le|x|<b_j} dF(x) \le \sum_{j=N+1}^{n} j \int_{b_{j-1}\le|x|<b_j} dF(x)$$

because $nb_j/b_n \le j$ for $j \le n$. We may now replace the n in the right hand side number above by ∞; as $N \to \infty$, it tends to 0 since

$$\sum_{k=1}^{\infty} k \int_{b_{k-1}\le|x|<b_k} dF(x) < \sum_{n=1}^{\infty} P(\|X_1\| \ge b_n) < \infty.$$

Thus we obtain (2) and this completes Lemma 2.

THEOREM. *Let $\{X_n : n = 1, 2, \cdots\}$ be a sequence of pairwise independent identically distributed random vectors taking values in a separable Banach space B. Assume that $E\|X_1\| = \infty$. Then for any sequence $\{b_n\}$ of positive numbers, only two cases are possible;*

(1) $\limsup_{n\to\infty} \|S_n\|/b_n = \infty$ *a.s. or*

(2) *there is a subsequence $\{n_k\}$ of the natural numbers such that $S_{n_k}/b_{n_k} \to 0$ a.s.*

Consequently, if we take a sequence $\{\gamma_n(\alpha)\}$, then for any α with $0 < \alpha < 1$, (1) holds.

PROOF. Suppose that (1) does not hold. Then we have

$$\limsup_{n\to\infty} b_n/n = \infty.$$

Define

$$\alpha_n = n \max\{b_1, b_2/2, \cdots, b_n/n\}.$$

Then the sequence $\{\alpha_n/n\}$ is nondecreasing, and satisfies $\alpha_n \geq b_n$ and $\alpha_{m_k} = b_{m_k}$ for an infinite subsequence $\{m_k\}$. By Lemma 2 there exists a further subsequence $\{n_k\}$ of subsequence $\{m_k\}$ such that $\|S_{n_k}\|/\alpha_{n_k} \to 0$ *a.s.*, which completes the proof.

References

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