Comm. Korean Math. Soc. 10 (1995), No. 2, pp. 419–438

DIFFUSION APPROXIMATION OF TIME DEPENDENT QUEUE SIZE DISTRIBUTION FOR $M^X/G^Y/c$ SYSTEM[1]

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ABSTRACT. We investigate a transient diffusion approximation of queue size distribution in $M^X/G^Y/c$ system using the diffusion process with elementary return boundary. We choose an appropriate diffusion process which approximates the queue size in the system and derive the transient solution of Kolmogorov forward equation of the diffusion process. We derive an approximation formula for the transient queue size distribution and mean queue size, and then obtain the stationary solution from the transient solution. Accuracy evaluation is presented by comparing approximation results for the mean queue size with the exact results or simulation results numerically.

1. Introduction

We deal with a transient diffusion approximation for multiserver bulk queue in which customers arrive in batches and are served in batches. Examples of bulk queueing models are an inventory model with demands of random size and replenishments of random quantity, a line of people who arrive in groups of random size waiting for an elevator, and the number of data units of random size which are generated at a remote terminal waiting for transmission to a central computer system. There are many analytical results for the bulk queueing model with either batch-arrival and single-service or single-arrival and batch-service (for the references see Chaudhry and Templeton [4]). Because of difficulties of analysis and the impracticability of analytical results, approximation or numerical methods are proposed for the multiserver bulk queue.

Received September 5, 1994.

AMS Subject Classification: 60J60, 60K25, 90B22.

Key Words: Bulk queue, Transient diffusion approximation, Queue size distribution, Elementary return boundary.

This paper was supported in part by NON DIRECTED RESEARCH FUND, KOREA Research Foundation, 1993.

Baguchi and Templeton [2] proposed a numerical method for evaluating the queue size distribution for $M^X/M^Y/1/K$ queue where X and Y are geometrically distributed. Baba [1] proposed a practical algorithm for computing the steady state probabilities of queue size for $M^X/PH/c$ system where the batch size X has a discerte phase type distribution. Recently Chakravarthy and Alfa [3] proposed a matrix-geometric algorithm for the computation of the steady state probability vectors and some performance measures of the multiserver queue with Markovian arrivals and batch services with thresholds. Zhao [23] gave an explicit expression of the generating function of equilibrium probabilities of the number of customers in the $GI^X/M/c$ system.

Diffusion approximations for single server bulk queue in steady-state were given by Chiamsiri and Leonard [5],Fisher [15], Gaver [16] and Gelenbe [17] etc. Kimura and Ohsone [21] gave a diffusion approximation for steady-state queue size distribution in $M^X/G/c$ system. Choi and Shin [8] suggested a transient diffusion approximation for the queue size distribution in $GI^X/G/1$ system. Recently the transient diffusion approximation for queue size distribution in multiserver system $M/G/c$, $M/G/c/N$ and $GI/G/c$ are also given by Choi and Shin [6, 7, 10]. The diffusion approximation for queue size distribution is based on the heavy traffic results that the suitably normalized queue size process can be approximated by Brownian motion process with reflecting boundary under the heavy traffic condition (Igelhart and Whitt [19]). It is known that a diffusion proces with elementary return boundary gives more accurate approximation for light traffic conditions in which the system is more frequently empty (Gelenbe [17]).

The analytical or algorithmic results for a transient distribution of the queue size for the multiserver bulk queue with both batch arrival and batch service are few.

The purpose of this paper is to provide a transient diffusion approximation of queue size distribution in $M^X/G^Y/c$ system using the diffusion process with elementary return boundary. In section 2 we choose an appropriate diffusion process which approximates the queue size in the system and derive the transient solution of Kolmogorov forward equation of the diffusion process. In section 3 we derive the approximation formula for the transient queue size distribution and mean queue size.

In section 4 we obtain the steady-state solution from the transient solution by letting $t \to \infty$. Accuracy evaluation is presented in section 5 by comparing approximation results for the mean queue size with the exact results or simulation results.

2. Formulation of diffusion process

For the $M^X/G^Y/c$ system we assume the following characteristics.

(i) Customers arrive in batches according to a time-homogeneous Poisson process with rate λ.

(ii) The batch size X of each arrival is positive integer valued random variable with probability mass function $x_k = P(X = k), k = 1, 2, \cdots$, and probability generating function $X(z) = \sum_{k=1}^{\infty} x_k z^k, |z| \leq 1$. We assume that $0 < \bar{x} = E(X) < \infty$ and $0 < \sigma_x^2 = E(X^2) - \bar{x}^2 < \infty$.

(iii) There are c identical servers acting in parallel and each server services in batches. The mean and variance of service time are $\frac{1}{\mu}$ and σ^2, respectively. The batch size Y of each server is a positive integer valued random variable with mean $0 < \bar{y} < \infty$ and variance $0 < \sigma_y^2 < \infty$.

(iv) The operation rules adopted here are as follows. When one of the servers is free,

(a) if the number of customers in queue is greater than Y, then the first Y customers enter service immediately,

(b) if the number of customers in queue is less than or equal to Y, then all the waiting customers are taken up for service,

(c) if the number of customers in queue equals to 0, service starts as soon as a new arrival occurs.

Let $Q(t)$ be the total number of customers in the $M^X/G^Y/c$ system at time t. To approximate the process $\{Q(t), t \geq 0\}$, we take a diffusion process $\{X_d(t), t \geq 0\}$ with state space $[0, \infty)$ and with elementary return boundary at the origin ([5-10],[13], [17], [18], [20], [21] etc.). The process $\{X_d(t), t \geq 0\}$ behaves as follows. When the trajectory of $X_d(t)$ reaches the boundary $x = 0$, it remains there for a random interval of time called a holding time. After the sojourn at the boundary the trajectory jumps into the interior $(0, \infty)$ and starts from scratch. In the queueing theoretic context the holding time at $x = 0$ corresponds to the time interval during which the system is empty. Since arrival process of customers is a compound Poisson, we assume the holding time distribution of $X_d(t)$ is exponential with rate λ. The diffusion process $\{X_d(t), t \geq 0\}$ is specified by two diffusion parameters called infinitesimal variance $a(x)$ and infinitesimal mean $b(x)$ defined by

$$a(x) = \lim_{\Delta t \to 0} \frac{Var(X_d(t+\Delta t) - X_d(t) | X_d(t) = x)}{\Delta t},$$
$$b(x) = \lim_{\Delta t \to 0} \frac{E(X_d(t+\Delta t) - X_d(t) | X_d(t) = x)}{\Delta t}.$$

Now we specify these parameters $a(x)$ and $b(x)$. Let $A(t)$ and $D(t)$ denote the total number of arriving customers and total number of departing customers in the time interval $(0,t)$, respectively. Then the number of customers in the system at time t is given by

$$Q(t) = Q(0) + A(t) - D(t).$$

Since $\{A(t), t \geq 0\}$ is a compound Poisson process, we have for large t

$$E(A(t+\Delta t) - A(t)) = \lambda \bar{x} \Delta t + o(\Delta t) \tag{2.1a}$$

$$\begin{aligned} Var(A(t+\Delta t) - A(t)) &= \lambda(\bar{x}^2 + \sigma_x^2)\Delta t + o(\Delta t) \\ &= \lambda \bar{x}^2 (1 + C_x^2)\Delta t + o(\Delta t), \end{aligned} \tag{2.1b}$$

where $C_x^2 = \frac{\sigma_x^2}{\bar{x}^2}$ is the square of coefficient of variation of the random variable X. Next we consider the departure process $D(t)$. Let $D_i(t)$ denote the total number of customers served by server i, $(i = 1, 2, \cdots, c)$ in the time interval $(0,t)$. Then we have $D(t) = \sum_{i=1}^{c} D_i(t)$. Let us assume the process $D_i(t)$ is cumulative process (see Cox [11])(this assumption seems to be appropriate under the heavy traffic condition $\rho = \frac{\lambda \bar{x}}{\mu c \bar{y}} \approx 1$) i.e.

$$D_i(t) = \sum_{j=1}^{N_d(t)} Y_j,$$

where Y_j are independent and identically distributed random variables with the same distribution as Y and $N_d(t)$ is the number of batch departures from the server i. Then $D_i(t)$ is asymptotically normal with mean $\mu\bar{y}t$ and variance $(C_s^2 + C_y^2)\mu\bar{y}^2 t$ as t goes to ∞ (eg. see Cox [11]), where $C_s^2 = \mu^2\sigma^2$ and $C_y^2 = \frac{\sigma_y^2}{\bar{y}^2}$ are the squares of coefficients of variation of service time and the random variable Y, respectively. If there are k customers in the system, then we may assume that $\min(c, \lceil\frac{k}{\bar{y}}\rceil)$ servers are busy, where $\lceil x\rceil$ denotes the smallest integer not smaller than x. From the discussion above, we have the asymptotic behavior of $D(t)$ as follows

$$E(D(t+\Delta t) - D(t)|Q(t) = k) \sim \min(c, \lceil\frac{k}{\bar{y}}\rceil)\bar{y}\mu\Delta t, \tag{2.2a}$$

$$Var(D(t+\Delta t) - D(t)|Q(t) = k) \sim \min(c, \lceil\frac{k}{\bar{y}}\rceil)(C_s^2 + C_y^2)\mu\bar{y}^2\Delta t. \tag{2.2b}$$

From (2.1) and (2.2) we choose the parameters depending on the state as follows for $k-1 < x \leq k, k = 1, 2\cdots$

$$a(x) = \lambda\bar{x}^2(1 + C_x^2) + \min(c, \lceil\frac{k}{\bar{y}}\rceil)(C_s^2 + C_y^2)\mu\bar{y}^2, \tag{2.3a}$$

$$b(x) = \lambda\bar{x} - \min(c, \lceil\frac{k}{\bar{y}}\rceil)\mu\bar{y}. \tag{2.3b}$$

When $c = 1$, i.e. single server system, the expression (2.3) is consistent with the parameters in Chiamsiri and Leonard [5] and if $\bar{y} = 1$, then the expression (2.3) is the same as the parameters in Kimura [21]. When $Y \equiv 1$ and $X \equiv 1$, i.e. $M/G/c$ system, the expression (2.3) is the same as the parameters in Kimura [20] and Choi and Shin [6]. Since the jump size of diffusion process at $x = 0$ corresponds to the arrival batch size X, we choose the jump distribution as $x_i = P\{X = i\}$. Then the probability density function $f(x, t|x_0)$ of $X_d(t)$ given $X_d(0) = x_0$ satisfies the following partial differential equation (Feller [14])

$$\begin{aligned} \frac{\partial f}{\partial t} =& \frac{1}{2}\frac{\partial^2}{\partial x^2}(a(x)f(x,t|x_0)) - \frac{\partial}{\partial x}(b(x)f(x,t|x_0)) \\ &+ \lambda P(t)\sum_{i=1}^{\infty}\delta(x-i)x_i, \ t > 0, \ x > 0 \end{aligned} \tag{2.4}$$

with the initial and boundary conditions

$$f(x,0|x_0)=\delta(x-x_0), \tag{2.5}$$

$$f(0,t|x_0)=0, \tag{2.6}$$

where $P(t)$ denotes the probability that the process $X_d(t)$ is at origin at time t and $\delta(\cdot)$ is Dirac's delta function. Furthermore, the probability $P(t)$ satisfies the following differential equation

$$\frac{dP(t)}{dt}=-\lambda P(t)+\lim_{x\to 0}\left[\frac{1}{2}\frac{\partial}{\partial x}(af)-bf\right] \tag{2.7}$$

and initial condition

$$P(0)=1(x_0=0), \tag{2.8}$$

where $1(E)$ denotes the indicator function of E. Since $X_d(t)$ approximates the number of customers in the system at time t, we assume the initial value $X(0)=x_0$ is nonnegative integer throughout this paper.

Now we derive the solution of the equation (2.4) under conditions (2.5) - (2.8). The parameters $a(x)$ and $b(x)$ in (2.3) are piecewise constants and have m discontinuous points, where $m=\min\{k|(c-1)\bar{y}<k, k=1,2,\cdots\}$. For the notational simplicity, let $a_k=a(k), b_k=b(k), k=1,2,\cdots,m$ and $f_k(x,t|x_0)$ be the restriction of $f(x,t|x_0)$ to $k-1<x\leq k, t\geq 0, k=1,2,\cdots,m-1$ and $f_m(x,t|x_0)$ the restriction of $f(x,t|x_0)$ to $m-1<x<\infty, t\geq 0$ and let $g_k(t|x_0)=f(k,t|x_0), k=0,1,2,\cdots,m-1$. Then the equation (2.4) is reduced to the following m equations. For $k-1<x<k, t\geq 0, k=1,2,\cdots,m-1$

$$\frac{\partial f_k}{\partial t}=\frac{1}{2}a_k\frac{\partial^2 f_k}{\partial x^2}-b_k\frac{\partial f_k}{\partial x}, \tag{2.9}$$

$$f_k(k-1,t|x_0)=g_{k-1}(t|x_0), \tag{2.10a}$$

$$f_k(k,t|x_0)=g_k(t|x_0), \tag{2.10b}$$

$$f_k(x,0|x_0)=\delta(x-x_0). \tag{2.10c}$$

For $m-1<x<\infty, t\geq 0$,

$$\frac{\partial f_m}{\partial t}=\frac{1}{2}a_m\frac{\partial^2 f_m}{\partial x^2}-b_m\frac{\partial f_m}{\partial x}+\lambda P(t)\sum_{i=m}^{\infty}\delta(x-i)x_i, \tag{2.11}$$

$$f_m(m-1,t|x_0)=g_{m-1}(t|x_0), \tag{2.12a}$$

$$f_m(x,0|x_0)=\delta(x-x_0). \tag{2.12b}$$

The Laplace transform solution $f^*(x,s|x_0)$ of $f(x,t|x_0)$ of the equation (2.4) is given as follows (for the similar derivation see appendix of Choi and Shin [6]). For $k-1<x<k, k=1,2,\cdots,m-1$,

$$\begin{aligned} f_k^*(x,s|x_0) =& \exp\big(\frac{b_k}{a_k}(x-k)\big)\frac{\sinh A_k(x-k+1)}{\sinh A_k}g_k^*(s|x_0) \\ &+\exp\big(\frac{b_k}{a_k}(x-k+1)\big)\frac{\sinh A_k(k-x)}{\sinh A_k}g_{k-1}^*(s|x_0). \end{aligned} \tag{2.13}$$

For $m-1<x<\infty$,

(2.14)

$$\begin{aligned} &f_m^*(x,s|x_0) \\ &=\exp\Big(\big(\frac{b_m}{a_m}-A_m\big)(x-m+1)\Big)g_{m-1}^*(s|x_0) \\ &\quad+\frac{1}{a_mA_m}e^{\frac{b_m}{a_m}(x-x_0)}\Big(e^{-A_m|x-x_0|}-e^{-A_m(x+x_0-2(m-1))}\Big)1(x_0\ge m) \\ &\quad+\frac{\lambda P^*(s)}{a_mA_m}\sum_{i=m}^{\infty}x_ie^{\frac{b_m}{a_m}(x-i)}\Big(e^{-A_m|x-i|}-e^{-A_m(x+i-2(m-1))}\Big) \\ &=\exp\Big(\big(\frac{b_m}{a_m}-A_m\big)(x-m+1)\Big)g_{m-1}^*(s|x_0) \\ &\quad+\frac{1}{a_mA_m}e^{\frac{b_m}{a_m}(x-x_0)}\Big(e^{-A_m|x-x_0|}-e^{-A_m(x+x_0-2(m-1))}\Big)1(x_0\ge m) \\ &\quad+\frac{\lambda P^*(s)}{a_mA_m}\Big(e^{\theta x}X(e^{-\theta};x) \\ &\qquad\qquad+\sum_{m\le i<x}x_ie^{\eta(x-i)}-e^{\eta x+2A_m(m-1)}(X(e^{-\theta};m)\Big), \end{aligned}$$

where $A_k=\frac{\sqrt{2a_ks+b_k^2}}{a_k}, k=1,2,\cdots,m,\ \theta=\frac{b_m}{a_m}+A_m,\ \eta=\frac{b_m}{a_m}-A_m$ and $X(z;k)=\sum_{i\ge k}x_iz^i$.

Let $z_1(s)=(\lambda+s)P^*(s)$ and $z_i(s)=g_{i-1}^*(s|x_0),\ i=2,3,\cdots,m$. Then the m-vector $\vec{z}(s)$ with components $z_i(s)$ satisfies the following linear system (detailed derivation is given in appendix A)

$$T(s)\vec{z}(s) = \vec{r}(s), \tag{2.15}$$

where $T(s)$ is the $m \times m$ matrix of the form

$$T(s) = \begin{pmatrix} q_1(s) & u_1(s) & 0 & 0 & \cdots & 0 & 0 \\ q_2(s) & v_2(s) & u_2(s) & 0 & \cdots & 0 & 0 \\ q_3(s) & w_3(s) & v_3(s) & u_3(s) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \ddots & \vdots \\ q_{m-1}(s) & 0 & \cdots & 0 & w_{m-1}(s) & v_{m-1}(s) & u_{m-1}(s) \\ q_m(s) & 0 & \cdots & 0 & 0 & w_m(s) & v_m(s) \end{pmatrix}$$

and the components of the matrix $T(s)$ and vector $\vec{r}(s)$ are as follows:

$$q_k(s) = \begin{cases} 1, & \text{if } k = 1 \\ -\frac{\lambda}{\lambda+s} x_{k-1}, & \text{if } k = 2, 3, \cdots, m-1 \\ -\frac{\lambda}{\lambda+s}\left(x_{m-1} + X(e^{(m-1)\theta}; m)\right), & \text{if } k = m, \end{cases}$$

$$u_k(s) = -B_k, k = 1, 2, \cdots, m-1,$$

$$v_k(s) = C_k, k = 2, 3, \cdots, m,$$

$$w_k(s) = -B_{k-1} \exp\left(2\frac{b_{k-1}}{a_{k-1}}\right), k = 3, 4, \cdots, m,$$

$$r_k(s) = \begin{cases} 1(x_0 = k-1), & \text{if } k = 1, 2, \cdots, m-1 \\ e^{-\theta(x_0-m+1)} 1(x_0 \geq m-1), & \text{if } k = m, \end{cases}$$

where

$$B_k = \frac{a_k A_k}{2} e^{-\frac{b_k}{a_k}} \frac{1}{\sinh A_k}, k = 1, 2, \cdots, m,$$

$$C_k = \frac{b_k - b_{k-1}}{2} + \frac{a_{k-1}A_{k-1}}{2} \frac{1}{\tanh A_{k-1}} + \frac{a_k A_k}{2} \frac{1}{\tanh A_k},$$
$$k = 2, 3, \cdots, m-1,$$

$$C_m = \frac{b_m - b_{m-1}}{2} + \frac{a_{m-1}A_{m-1}}{2} \frac{1}{\tanh A_{m-1}} + \frac{a_m A_m}{2}.$$

3. Diffusion approximation of time dependent queue size distribution for $M^X/G^Y/c$ system

Now we derive approximation formulas of probability function $p(k, t|x_0) = P(Q(t) = k|Q(0) = x_0)$ and mean $L(t|x_0) = E(Q(t)|Q(0) = x_0)$. Let $\hat{p}(k,t|x_0)$, $(k = 0,1,2,\cdots)$ be an approximation of $p(k,t|x_0)$, $(k = 0,1,2,\cdots)$. From the definition of the elementary return boundary, it seems to be appropriate to use

$$\hat{p}(0,t|x_0) = P(t). \tag{3.1}$$

To obtain an approximation $\hat{p}(k,t|x_0)$, $(k = 1,2,\cdots)$, we must discretize the probability density function $f(x,t|x_0)$. Although there are many discretization methods (e.g. see Gelenbe, Pusolle and Nelson [18]), we adopt the following, because of its computational simplicity and high accuracy

$$\hat{p}(k,t|x_0) = D(t)f(k,t|x_0), \;\; k = 1,2,\cdots \tag{3.2}$$

where $D(t)$ is the normalizing constant so as to be $\sum_{k=0}^{\infty} \hat{p}_1(k,t|x_0) = 1$, that is,

$$D(t) = \frac{1-P(t)}{\sum_{k=1}^{\infty} f(k,t|x_0)}.$$

We have from (2.13) and (2.14) the Laplace transform $f^*(k,s|x_0)$ of $f(k,t|x_0)$ as follows

$$f^*(k,s|x_0) = g_k^*(s|x_0), \;\; k = 1,2,\cdots,m-1, \tag{3.3a}$$

(3.3b)

$$\begin{aligned} f^*(k,s|x_0) &= e^{\eta(k-m+1)} g_{m-1}^*(s|x_0) \\ &+ \frac{1}{a_m A_m} e^{\frac{b_m}{a_m}(k-x_0)} \Big(e^{-A_m|k-x_0|} - e^{-A_m(k+x_0-2(m-1))} \Big) 1 \quad (x_0 \geq m) \\ &+ \frac{\lambda P^*(s)}{a_m A_m} \Big(e^{k\theta} X(e^{-\theta};k) + \sum_{i=m}^{k-1} x_i e^{(k-i)\eta} \\ &\qquad - e^{k\eta + 2A_m(m-1)} X(e^{-\theta};m) \Big), \qquad k \geq m. \end{aligned}$$

To obtain the numerical results of $\hat{p}(k,t|x_0)$, it is necessary to reduce the infinite sum $\sum_{k=1}^{\infty} f(k,t|x_0)$ to the computable form. Let $D_0(t) = \sum_{k=1}^{\infty} f(k,t|x_0)$. We have from (3.3) that the Laplace transform $D_0^*(s)$ of $D_0(t)$

$$
\begin{aligned}
D_0^*(s) &= \sum_{k=1}^{\infty} f^*(k,s|x_0) \\
&= \sum_{k=1}^{m-1} g_k^*(s|x_0) + \frac{e^{\eta}}{1-e^{\eta}} g_{m-1}^*(s|x_0) \\
&\quad + \frac{1}{a_m A_m}\Big(\frac{1-e^{-2A_m-\theta(x_0-m)}}{1-e^{\eta}} - \frac{1-e^{-\theta(x_0-m)}}{1-e^{\theta}}\Big) 1(x_0 \geq m) \\
&\quad + \frac{\lambda P^*(s)}{a_m A_m}\Big(\frac{\bar{x}_m - e^{-2A_m} e^{m\theta} X(e^{-\theta};m)}{1-e^{\eta}} - \frac{\bar{x}_m - e^{m\theta} X(e^{-\theta};m)}{1-e^{\theta}}\Big).
\end{aligned} \tag{3.4}
$$

Now we derive approximation $\hat{L}(t|x_0)$ of mean number $L(t|x_0)$ of customers in the system by using the formula

$$
\hat{L}(t|x_0) = \sum_{k=1}^{\infty} k\hat{p}(k,t|x_0) = D(t)L_0(t|x_0), \tag{3.5}
$$

where $L_0(t|x_0) = \sum_{k=1}^{\infty} kf(k,t|x_0)$.

The Laplace transform $L_0^*(s|x_0)$ of $L_0(t|x_0)$ is given as follows

$$
\begin{aligned}
L_0^*(s|x_0) &= \sum_{k=1}^{m-1} kg_k^*(s|x_0) + g_{m-1}^*(s|x_0)\frac{e^{\eta}(m-(m-1)e^{\eta})}{(1-e^{\eta})^2} \\
&\quad + \frac{1}{a_m A_m}\Big(\frac{1}{(1-e^{\theta})^2}\left[(m-(m-1)e^{\theta})e^{-\theta(x_0-m)} - (x_0-(x_0-1)e^{\theta})\right] \\
&\quad + \frac{x_0(1-e^{\eta})+e^{\eta}}{(1-e^{\eta})^2} - e^{-\theta(x_0-m)-2A_m}\frac{m-(m-1)e^{\eta}}{(1-e^{\eta})^2}\Big) \; 1(x_0 \geq m) \\
&\quad + \frac{\lambda P^*(s)}{a_m A_m}\Big(\frac{1}{(1-e^{\theta})^2}\Big[(m-(m-1)e^{\theta})e^{m\theta}X(e^{-\theta};m) \\
&\qquad - (1-e^{\theta})(\bar{x} - \sum_{k=1}^{m-1} kx_k) - e^{\theta}\bar{x}_m\Big] \\
&\quad + \frac{1}{1-e^{\eta}}(\bar{x} - \sum_{k=1}^{m-1} kx_k) + \frac{e^{\eta}}{(1-e^{\eta})^2}\bar{x}_m \\
&\qquad - e^{-2A_m}\frac{m-(m-1)e^{\eta}}{(1-e^{\eta})^2}e^{m\theta}X(e^{-\theta};m)\Big).
\end{aligned} \tag{3.6}
$$

4. Diffusion approximation of steady-state queue size distribution for $M^X/G^Y/c$ system

In this section we derive a diffusion approximation of steady-state queue size distribution. For this we first derive the stationary probability density function of the diffusion process $\{X(t), t \geq 0\}$ from the transient density function by using the final value theorem for Laplace transform: $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sf^*(s)$. Let $f(x) = \lim_{t\to\infty} f(x,t|x_0)$ and $P = \lim_{t\to\infty} P(t)$. Under the condition $b_m < 0$, that is, $\rho = \frac{\lambda \bar{x}}{c\mu \bar{y}} < 1$, the stationary density function is given as follows (the derivation is given in the Appendix). For the notational simplicity, let $\bar{x}_k = P(X \geq k) = \sum_{j=k}^{\infty} x_j$ and $\gamma_k = \frac{2b_k}{a_k}$ $(k = 1, 2, \cdots, m)$, and

$$B_j(0) = \lim_{s\to 0} B_j(s) = \begin{cases} \frac{b_j}{e^{\gamma_j}-1} & \text{if } b_j \neq 0 \\ \frac{a_k}{2} & \text{if } b_j = 0, \end{cases}$$

$$q_k = \sum_{j=1}^{k} \frac{\bar{x}_j}{B_j(0)} \exp(\sum_{i=j+1}^{k} \gamma_i), \; k = 1, 2, \cdots, m-1.$$

When $b_k \neq 0, k = 1, 2, \cdots, m$, we have for $k-1 < x \leq k$

(4.1)
$$f(x) = \begin{cases} \lambda P\Big(e^{\gamma_k(x-k)}(q_k + \frac{\bar{x}_k}{b_k}) - \frac{\bar{x}_k}{b_k}\Big), & \text{if } k = 1, 2, \cdots, m-1 \\ \lambda P\Big(e^{\gamma_m(x-m+1)}(q_{m-1} + \frac{\bar{x}_m}{b_m}) \\ -\frac{1}{b_m}\big(\bar{x}_k + \sum_{i=m}^{k-1} e^{\gamma_m(x-i)} x_i\big)\Big), & \text{if } k = m, m+1, \cdots, \end{cases}$$

(4.2)
$$P = \Big[1 + \lambda \sum_{k=1}^{m-1} \Big(\frac{1-e^{-\gamma_k}}{\gamma_k}(q_k + \frac{\bar{x}_k}{b_k}) - \frac{\bar{x}_k}{b_k}\Big) - \lambda \frac{q_{m-1}}{\gamma_m} - \frac{\lambda}{b_m}(\bar{x} - \sum_{k=1}^{m-1} k x_k - (m-1)\bar{x}_m)\Big]^{-1}$$

and when $b_i = 0$ for some $i = 1, 2, \cdots, m-1$, for $i - 1 \leq x < i$,

(4.3)
$$f(x) = \lambda P(q_i + \frac{2}{a_i} \bar{x}_i (x - i))$$

$$P = \Big[1 + \lambda \sum_{\substack{k=1 \\ k\neq i}}^{m-1} \Big(\frac{1-e^{-\gamma_k}}{\gamma_k}\big(q_k + \frac{\bar{x}_k}{b_k}\big) - \frac{\bar{x}_k}{b_k}\Big) + \lambda\big(q_i - \frac{\bar{x}_i}{a_i}\big) \tag{4.4}$$
$$- \lambda \frac{q_{m-1}}{\gamma_m} - \lambda \frac{1}{b_m}\big(\bar{x} - \sum_{k=1}^{m-1} k x_k - (m-1)\bar{x}_m\big)\Big]^{-1}.$$

We have from (3.1), (3.2), (4.1) and (4.2) that the steady-state probability $\hat{p}(k) = \lim_{t\to\infty} \hat{p}(k,t|x_0)$

(4.5)

$$\hat{p}(k) = \begin{cases} P, & \text{if } k = 0 \\ G q_k, & \text{if } k = 1, 2, \cdots, m-1 \\ G\Big(e^{\gamma_m(k-m+1)}\big(q_{m-1} + \frac{\bar{x}_m}{b_m}\big) & \\ -\frac{1}{b_m}\big(\bar{x}_k + \sum_{i=m}^{k-1} x_i e^{\gamma_m(k-i)}\big)\Big), & \text{if } k \geq m, \end{cases}$$

where

$$G = \lambda P \frac{1-P}{\sum_{k=1}^{\infty} f(k)} = (1-P) \tag{4.6}$$
$$\times \Big(\sum_{k=1}^{m-2} q_k + \frac{q_{m-1}}{1-e^{\gamma_m}} - \frac{1}{b_m}\big(\bar{x} - \sum_{k=1}^{m-1} k x_k + (m-1)\bar{x}_m\big)\Big)^{-1}.$$

The stationary mean queue size $\hat{L} = \lim_{t\to\infty} \hat{L}(t|x_0)$ is given by

$$\hat{L} = G\sum_{k=1}^{m-1} k q_k + \frac{G}{b_m}\Big[\frac{e^{\gamma_m}}{(1-e^{\gamma_m})^2}(m - (m-1)e^{\gamma_m}) b_m q_{m-1} \tag{4.7}$$
$$- \frac{e^{\gamma_m}}{1-e^{\gamma_m}}(\bar{x} - (m-1)\bar{x}_m - \sum_{k=1}^{m-1} k x_k) - \frac{1}{2}\bar{x}^2(C_x^2 + 1 + \frac{1}{\bar{x}}) + \sum_{k=1}^{m-1} k\bar{x}_k\Big].$$

5. Numerical Results

In this section we investigate the accuracy of approximation by numerically comparing the diffusion approximation results with the simulation results or analytic results for the mean queue size. We use three service time distributions with mean 1 which are exponential (denoted by M), hyperexponential of order 2 (denoted by H_2) and 2 stage Erlang (denoted by E_2). As the batch size distribution we use the geometric distribution with mean 2 (denoted by $G(2)$), deterministic distribution with mean k (denoted by $D(k)$), $k = 1, 2$ and uniform distribution on $\{1, 2, 3\}$. In table 1, we compare the approximation results with the analytic results for stationary mean queue size in $M^X/G/c$ system for the traffic intensities $\rho = 0.3, 0.5, 0.7, 0.9$. In table 1, the relative percentage error Err(%) denotes the

$$\text{Err}(\%) = \frac{\text{Exact - Diff.}}{\text{Exact}} \times 100.$$

We cite the tables in Kimura [20,21] for the analytic results. Table 2 represents the comparisons of diffusion approximation results with the simulation results for the mean queue size in $M^{G(2)}/G^{U(2)}/10$ system for moderate traffic intensity $\rho = 0.5$. In table 2 "sim." and "c.i." denote the simulation results and 95 % confidence interval, respectively. The numerical inversion of Laplace transform $\hat{L}^*(s|x_0)$ is obtained by using the algorithm 368 in ACM [22]. Table 1 and table 2 show that our approximation gives a good numerical results.

Table 1. Mean Queue Size for Stationary $M^X/G/10$ System

	X	$G(2)$		$D(2)$		$D(1)$	
G	ρ	Exact	L (Err(%))	Exact	L (Err(%))	Exact	L (Err(%))
	0.3	3.043	3.020 (0.756)	3.006	3.013 (-0.233)	3.000	3.002 (0.067)
M	0.5	5.355	5.211 (2.689)	5.138	5.107 (0.603)	5.036	5.032 (0.079)
	0.7	8.956	8.627 (3.674)	8.129	8.026 (1.267)	7.517	7.510 (0.093)
	0.9	23.021	22.489 (2.311)	18.875	18.680 (1.033)	15.018	14.999 (0.127)
	0.3	3.038	3.223 (-6.090)	3.006	3.228 (-7.385)	3.001	3.239 (-7.931)
H_2	0.5	5.343	5.534 (-3.575)	5.141	5.419 (-5.408)	5.043	5.329 (-5.671)
	0.7	9.027	9.271 (-2.703)	8.241	8.639 (-4.830)	7.637	8.070 (-5.670)
	0.9	24.40	24.802 (-1.648)	20.35	20.950 (-2.948)	16.50	17.196 (-4.218)
	0.3	3.054	2.822 (7.597)	3.008	2.800 (6.915)	3.000	2.765 (7.833)
E_2	0.5	5.381	4.904 (8.865)	5.137	4.813 (6.307)	5.029	4.757 (5.409)
	0.7	8.889	8.002 (9.979)	8.019	7.443 (7.138)	7.407	7.002 (5.468)
	0.9	21.60	20.195 (6.505)	17.40	16.442 (5.506)	13.576	12.863 (5.252)

Table 2. Mean Queue Size for $M^{G(2)}/G^{U(2)}/10$ System ($\rho = 0.5, x_0 = 0$)

	M		E_2		H_2	
Time	Sim (c.i.)	$L(t)$	Sim (c.i.)	$L(t)$	Sim (c.i.)	$L(t)$
.10	.976 (.034)	.956	.966 (.034)	.948	1.018 (.035)	.964
.30	2.614 (.053)	2.610	2.563 (.053)	2.559	2.895 (.058)	2.660
.50	3.994 (.065)	3.966	3.856 (.064)	3.861	4.549 (.072)	4.068
.70	5.035 (.072)	5.079	4.809 (.070)	4.922	5.816 (.080)	5.233
1.00	6.319 (.080)	6.401	5.598 (.078)	6.171	7.288 (.089)	6.626
3.00	9.872 (.104)	9.724	9.258 (.098)	9.243	10.318 (.107)	10.212
5.00	10.579 (.112)	10.242	10.250 (.109)	9.684	10.623 (.111)	10.823
7.00	10.631 (.113)	10.316	10.516 (.110)	9.735	10.614 (.111)	10.921
10.00	10.720 (.113)	10.345	10.716 (.114)	9.757	10.703 (.112)	10.971
15.00	10.786 (.115)	10.335	10.820 (.117)	9.739	10.723 (.114)	10.975
20.00	10.704 (.114)	10.335	10.752 (.116)	9.749	10.688 (.113)	10.972
30.00	10.648 (.115)	10.343	10.650 (.115)	9.748	10.609 (.113)	10.970

Appendix 1. Derivation of (2.15)

Taking the Laplace transform with respective to t-variable of the equation (2.7), we have the following relations

$$[C_{x,s}f_1^*]_{x\downarrow 0} = (\lambda + s)P^*(s) - 1(x_0 = 0) \tag{A.1}$$

where

$$C_{x,s}f^* = \frac{1}{2}\frac{\partial}{\partial x}\{a(x)f^*(x,t|x_0)\} - b(x)f^*(x,s|x_0).$$

To determine $g_k^*(s|x_0)$'s and $P^*(s)$ in terms of known parameters, we take the Laplace transform of equation (2.4) with respect to t-variable, and then integrate with respect to x variable. Then we have

$$\begin{aligned} C_{x,s}f^* =&[C_{x,s}f_1^*]_{x\downarrow 0} + s\int_0^x f^*(y,s|x_0)dy - 1(x \geq x_0) \\ &- \lambda P^*(s)\sum_{1\leq i\leq r} 1(x \geq i)x_i. \end{aligned} \tag{A.2}$$

After simple calculation we have from (A.2) that

$$[C_{x,s}f_2^*]_{x\downarrow 1} = [C_{x,s}f_1^*]_{x\uparrow 1} - \lambda P^*(s)x_1 - 1(x_0 = 1), \tag{A.3}$$

$$\begin{aligned}[C_{x,s}f_k^*]_{x\downarrow k-1} =& [C_{x,s}f_{k-1}^*]_{x\uparrow k-1} - \lambda P^*(s)x_{k-1} \\ &- 1(x_0 = k-1), k = 3,4,\cdots,m. \end{aligned} \tag{A.4}$$

Calculating $C_{x,s}f_1^*$ from (2.13) and then substituting it into (A.1), we have

$$(\lambda + s)P^*(s) - B_1 g_1^*(s|x_0) = 1(x_0 = 0). \tag{A.5}$$

Similarly we obtain by calculating $C_{x,s}f_k^*$ from (2.13) and substituting it into (A.3),

$$-\lambda x_1 P^*(s) + C_2 g_1^*(s|x_0) - B_2 g_2^*(s|x_0) = 1(x_0 = 1). \tag{A.6}$$

Doing the same thing for (A.3) and (A.4), from (2.13) and (2.14) yields the followings

(A.7)

$$\begin{aligned} &- \lambda P^*(s)x_{k-1} - B_{k-1}e^{2\frac{b_{k-1}}{a_{k-1}}} g_{k-2}^*(s|x_0) + C_k g_{k-1}^*(s|x_0) - B_k g_k^*(s|x_0) \\ =&1(x_0 = k-1), \quad k = 3, 4, \cdots, m-1, \\ &- \lambda P^*(s)\Big(x_{m-1} + e^{\theta(m-1)} \sum_{i=m}^{\infty} x_i e^{-i\theta}\Big) \end{aligned}$$

$$\begin{aligned} &- g_{m-2}^*(s|x_0)B_{m-1}e^{2\frac{b_{m-1}}{a_{m-1}}} + g_{m-1}^*(s|x_0)C_m \\ =&e^{-\theta(x_0-m+1)}1(x_0 \geq m-1). \end{aligned} \tag{A.8}$$

Summarizing the (A.5) - (A.8), we have the linear system (2.15).

2. Derivation of (4.1) and (4.2).

For the notational simplicity, let $\gamma_k = \frac{2b_k}{a_k}, k = 1, 2, \cdots, m$. Simple calculation yields

$$\lim_{s \to 0} A_k = \frac{|b_k|}{a_k}, \; k = 1, 2, \cdots, m, \tag{A.9}$$

$$\lim_{s \to 0} B_k = \begin{cases} \frac{b_k}{e^{\gamma_k} - 1} & \text{if } b_k \neq 0 \\ \frac{a_k}{2} & \text{if } b_k = 0 \end{cases} \quad k = 1, 2, \cdots, m-1, \tag{A.10}$$

$$\text{(A.11)} \qquad \lim_{s\to 0} C_k = \begin{cases} B_{k-1}(0) + e^{\gamma_k} B_k(0), & \text{if } k = 2, 3, \cdots, m-1 \\ B_{m-1}(0) & \text{if } k = m. \end{cases}$$

Multiplying both sides of (A.5) by s and letting $s \to 0$, we have

$$\text{(A.12)} \qquad g_1 = \frac{\lambda P}{B_1(0)} = \begin{cases} \frac{1}{b_1}(e^{\gamma_1} - 1)\lambda P & \text{if } b_1 \neq 0 \\ \frac{2}{a_1}\lambda P & \text{if } b_1 = 0. \end{cases}$$

Similarly from (A.6), we have

$$\text{(A.13)} \qquad -\lambda x_1 P + C_2(0) g_1 - B_2(0) g_2 = 0.$$

From (A.11) - (A.13) we have the relation for g_2 as

$$\text{(A.14)} \qquad \begin{aligned} g_2 &= \frac{1 - x_1}{B_2(0)}\lambda P + e^{\gamma_2} g_1 \\ &= \lambda P\Big(\frac{1 - x_1}{B_2(0)} + \frac{e^{\gamma_2}}{B_1(0)}\Big). \end{aligned}$$

Doing the same thing for (A.7) we have the following
(A.15)

$$\begin{aligned} g_k &= \frac{1}{B_k(0)}\Big(-x_{k-1}\lambda P - B_{k-1}(0) e^{\gamma_{k-1}} g_{k-2} + C_k(0) g_{k-1}\Big) \\ &= \frac{1}{B_k(0)}\Big(-x_{k-1}\lambda P - B_{k-1}(0)(g_{k-1} - e^{\gamma_{k-1}} g_{k-2})\Big) + e^{\gamma_k} g_{k-1}, \\ &\quad k = 3, 4, \cdots, m-1. \end{aligned}$$

Now we derive the concrete form from the recursive formula (A.15). For $k = 3$, we have from (A.14) and (A.15)

$$\begin{aligned} g_3 &= \frac{1}{B_3(0)}\Big(-x_2\lambda P + B_2(0)(g_2 - e^{\gamma_2} g_1)\Big) + e^{\gamma_3} g_2 \\ &= \frac{1}{B_3(0)}(-x_2\lambda P + (1 - x_1)\lambda P) + e^{\gamma_3} g_2 \\ &= \frac{1}{B_3(0)}(1 - x_1 - x_2)\lambda P + e^{\gamma_3} g_2. \end{aligned}$$

Thus we have inductively

$$\text{(A.16)} \qquad g_k - e^{\gamma_k} g_{k-1} = \frac{1}{B_k(0)}(1 - \sum_{i=1}^{k-1} x_i)\lambda P, \ k = 1, 2, \cdots, m-1.$$

Hence from (A.15) and (A.16) we have

$$\text{(A.17)} \qquad g_k = \frac{\lambda P}{B_k(0)} \bar{x}_k + e^{\gamma_k} g_{k-1}, k = 1, 2, \cdots, m-1,$$

where $\bar{x}_k = 1 - \sum_{i=1}^{k-1} x_i = P(X \geq k)$. From the recursive relation (A.17) and (A.12) we have

$$\text{(A.18)} \qquad g_k = \lambda P \sum_{j=1}^{k} \frac{\bar{x}_j}{B_j(0)} \exp(\sum_{i=j+1}^{k} \gamma_i), \ k = 1, 2, \cdots, m-1.$$

Now we calculate $\lim_{s \to 0} s f_k^*(x, s|x_0) = f_k(x), k = 1, 2, \cdots, m$. Routine calculation for $\lim_{s \to 0} s f_k^*(x, s|x_0) = f_k(x)$ yields the following for $k = 1, 2, \cdots, m-1$,

$$\text{(A.19)} \qquad \begin{aligned} f_k(x) &= e^{\gamma_k(x-k)} g_k + \frac{\bar{x}_k}{B_k(0)} \frac{e^{\gamma_k(x-k)} - 1}{e^{\gamma_k} - 1} \lambda P \\ &= \begin{cases} \lambda P\Big((q_k + \frac{\bar{x}_k}{b_k}) e^{\gamma_k(x-k)} - \frac{\bar{x}_k}{b_k}\Big) & \text{if } b_k \neq 0 \\ \lambda P\Big(q_k + \frac{2}{a_k} \bar{x}_k (x-k)\Big) & \text{if } b_k = 0, \end{cases} \end{aligned}$$

where $q_k = \frac{g_k}{\lambda P}, k = 1, 2, \cdots, m-1$, and
(A.20)

$$\begin{aligned} f_m(x) &= e^{\gamma_m(x-m+1)} g_{m-1} \\ &\quad - \frac{\lambda P}{b_m}\Big(\bar{x}_{\lceil x \rceil} - \bar{x}_m e^{\gamma_m(x-m+1)} + \sum_{m \leq i < x} e^{\gamma_m(x-i)} x_i\Big) \\ &= \lambda P\Big(e^{\gamma_m(x-m+1)}(q_{m-1} + \frac{\bar{x}_m}{b_m}) - \frac{1}{b_m}(\bar{x}_{\lceil x \rceil} + \sum_{m \leq i < x} e^{\gamma_m(x-i)} x_i)\Big). \end{aligned}$$

Now we calculate the $\lim_{t\to\infty} P(t) = P$ from the relation

$$P + \int_0^\infty f(x)dx = 1. \tag{A.21}$$

Simple calculation yields the followings. For $k = 1, 2, \cdots, m-1$,

$$\int_{k-1}^{k} f(x)dx = \begin{cases} \lambda P\Big((q_k + \frac{\bar{x}_k}{b_k})\frac{1-e^{-\gamma_k}}{\gamma_k} - \frac{\bar{x}_k}{b_k}\Big) & \text{if } b_k \neq 0 \\ \lambda P(q_k - \frac{\bar{x}_k}{a_k}) & \text{if } b_k = 0 \end{cases} \tag{A.22}$$

and

$$\int_{m-1}^{\infty} f(x)dx = -\lambda P \frac{1}{\gamma_m}(q_{m-1} + \frac{\bar{x}_m}{b_m}) - \frac{\lambda P}{b_m}\sum_{j=m}^{\infty}\Big(\bar{x}_j + \frac{1}{\gamma_m}\sum_{i=m}^{j-1} x_i(e^{\gamma_m(j-i)} - e^{\gamma_m(j-i-1)})\Big).$$

Note that

$$\sum_{j=m}^{\infty}\sum_{i=m}^{j-1} x_i(e^{\gamma_m(j-i)} - e^{\gamma_m(j-i-1)}) =(e^{\gamma_m} - 1)\sum_{j=m}^{\infty}\sum_{i=m}^{j-1} x_i e^{\gamma_m(j-i-1)} = -\bar{x}_m,$$

$$\sum_{k=m}^{\infty} \bar{x}_k = \bar{x} - m + 1 - \sum_{j=1}^{m-1}(j - m + 1)x_j.$$

Hence we have

$$\int_{m-1}^{\infty} f(x)dx = -\lambda P\Big(\frac{q_{m-1}}{\gamma_m} + \frac{1}{b_m}(\bar{x} - m + 1 - \sum_{j=1}^{m-1}(j - m + 1)x_j)\Big). \tag{A.23}$$

When $b_k \neq 0, k = 1, 2, \cdots, m-1$, we have from (A.22), (A.23) and (A.21) that

$$P = \Big[1 + \lambda\sum_{k=1}^{m-1}((q_k + \frac{\bar{x}_k}{b_k})\frac{1-e^{-\gamma_k}}{\gamma_k} - \frac{\bar{x}_k}{b_k}) - \lambda(\frac{q_{m-1}}{\gamma_m} + \frac{1}{b_m}(\bar{x} - m + 1 - \sum_{j=1}^{m-1}(j - m + 1)x_j))\Big]^{-1} \tag{A.24}$$

and when $b_i = 0$ for some $i = 1, 2, \cdots, m-1$, by letting $b_i \to 0$ in (A.24) we have

$$P = \Big[1 + \lambda \sum_{\substack{k=1 \\ k \neq i}}^{m-1} \big((q_k + \frac{\bar{x}_k}{b_k})\frac{1 - e^{-\gamma_k}}{\gamma_k} - \frac{\bar{x}_k}{b_k}\big) + \lambda(q_i - \frac{\bar{x}_i}{a_i}) - \lambda\frac{q_{m-1}}{\gamma_m} + \frac{\lambda}{b_m}(\bar{x} - m + 1 - \sum_{j=1}^{m-1}(j - m + 1)x_j))\Big]^{-1}.$$

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