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GEOMETRIC ERGODICITY AND TRANSIENCE FOR NONLINEAR AUTOREGRESSIVE MODELS

OESOOK LEE

ABSTRACT. we consider the R^k-valued ($k \geq 1$) processes $\{X_n\}$ generated by $X_n + 1 = f(X_n) + e_{n+1}$, where $f(x) = (h(x), x^{(1)}, \cdots, x^{(k-1)})'$. We assume that h is a real-valued measuable function on R^k and that $e_n = (e'_n, 0, \cdots, 0)'$ where $\{e'_n\}$ are independent and identically distributed random variables. We obtained a practical criteria guaranteeing a given process to be geometrically ergodic. Sufficient condition for transience is also given.

1. Introduction and preliminaries

Recently the class of nonlinear autoregressive models for discrete time series has been studied extensively (Nummelin[5], Suba Rao and Gabr[8], Tong[10], Priestley[11]). Many of these models are Markov chains or can be rephrased as Markov chains. The aim of this paper is to obtain conditions for the geometric ergodicity and also for the transience of such processes.

We let $\{X_n : n \geq 0\}$ be a homogeneous Markov chain taking values in $(S, \mathcal{S})$, where S is a set and $\mathcal{S}$ is a countably generated σ-algebra of subsets of S. We denote the *transition probabilities* for

$$P^n(x, A) = Pr(X_n \in A | X_0 = x), \qquad x \in S, \ A \in \mathcal{S},$$

with $P^1(x, A) = P(x, A)$.

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The Markov chain is $\varphi - irreducible$ if, for some σ-finite measure φ on $(S, \mathcal{S})$,

$$\sum_n P^n(x, A) > 0$$

for all $x \in S$, whenever $\varphi(A) > 0$.

A φ-irreducible chain is *recurrent* if for every $x \in S$ and every $A \in \mathcal{S}$ with $\varphi(A) > 0$,

$$\sum_n^{\infty} P^n(x, A) = \infty.$$

If the $\{X_n\}$ is nonrecurrent, we call $\{X_n\}$ *transient.*

If $\{X_n\}$ is φ-irreducible and recurrent, then there is a unique (up to constant multiples) nontrivil σ-finite measure μ satisfying

$$\mu(A) = \int_S \mu(dy)P(y, A), \qquad A \in \mathcal{S}.$$

This unique solution is stronger than φ and if it is finite, then we call $\{X_n\}$ *positive recurrent*, and denote the unique finite invariant measure by π.

It is known that if there is a finite invariant measure for $\{X_n\}$, then for π-a. e. x,

$$\|P^n(x, \cdot) - \pi\| \to 0, \quad \text{as} \quad n \to \infty,$$

where $\|\cdot\|$ denotes the total variation norm.

If $\{X_n\}$ is φ-irreducible, then the state space S is indecomposable and therefore $\{X_n : n \geq 0\}$ with invariant initial distribution π is an ergodic stationary prcess in the sence that the shift invariant σ-field is trivial under P_π (Nummelin [5]). However throughout this paper we shall call $\{X_n\}$ *ergodic* if there exists a probability measure π on $(S, \mathcal{S})$ such that

$$\lim_{n \to \infty} \|P^n(x, \cdot) - \pi\| = 0$$

for all $x \in S$.

If $\{X_n\}$ is ergodic and there exists ρ, $0 < \rho < 1$ such that

$$\lim_{n \to \infty} \rho^{-n} \|P^n(x, \cdot) - \pi(\cdot)\| = 0$$

for all $x \in S$, then $\{X_n\}$ is said to be *geometrically ergodic.*

To state the general criteria for checking the geometric ergodic of $\{X_n\}$, we need the concept of a small set. Let $\{X_n\}$ be a φ-irreducible Markov chain, call a set $B \in \varphi$ small if $\varphi(B) > 0$ and for every $A \in \varphi$ with $\varphi(A) > 0$, there exists a positive integer m such that

$$\inf_{x \in B} \sum_{n=1}^{m} P^n(x, A) > 0.$$

Let $\{X_n\}$ be an aperiodic Markov chain, and k a fixed integer. The following is proved easily (see Tjostheim, [9]): If $\{X_{nk}\}$ is recurrent (ergodic, geometrically ergodic, transient respectively), then so is $\{X_n\}$.

If we combine the above results with Tweedie's criteria (Tweedie [12], Nummelin and Tuominen [6]), we have the following k-step criteria:

For next two theorems, we assume that $\{X_n\}$ is a aperiodic, φ-irreducible chain, and let g be a nonnegative measurable function on S.

THEOREM 1.1. *$\{X_n\}$ is geometric ergodic if there exist a positive integer k, a small set $B, \varepsilon > 0$, $M < \infty$ and $r > 1$ such that*

$$E[g(X_{n+k})|X_n = x] \leq \frac{1}{r} g(x) - \varepsilon, \qquad x \in B^c$$
$$E[g(X_{n+k}) I_{\{X_{n+k} \in B^C\}} | X_n = x] \leq M, \qquad x \in B.$$

THEOREM 1.2. *$\{X_n\}$ is transient if there exists a positive integer k, and a set B such that both B and B^c have positive φ-measure, and*

$$E[g(X_{n+k})|X_n = x] \leq g(x), \quad x \in B^c$$
$$g(x) < \inf_{y \in B} g(y), \qquad x \in B^c.$$

In this paper, we are interested in R^k-valued ($k \geq 1$) processes $\{X_n\}$ generated by

$$X_{n+1} = f(X_n) + e_{n+1}, \qquad n \geq 0 \tag{1}$$

where $f(x) = (h(x), x^{(1)}, \cdots, x^{(k-1)})'$, for $x = (x^{(1)}, x^{(2)}, \cdots, x^{(k)})'$. We assume that h is a real-valued measurable function on R^k and that $e_n = (e'_n, 0, \cdots, 0)'$ where $\{e'_n\}$ are independent, identically distributed random variables, each having an absolutely continuous distribution with probability density function $q(\cdot)$. Take X_0 arbitrary but independent of e'_n, $n \geq 1$. $\{X_n\}$ generated by (1) can be thought as obtained by vectorization of the kth-order scalar equation

$$Z_n = h(Z_{n-1}, Z_{n-2}, \cdots, Z_{n-k}) + e'_n$$

with

$$X_n = (Z_n, Z_{n-1}, \cdots, Z_{n-k+1}).$$

Various models of this type have been studied by many authors such as Bhattacharya and Lee [1], Chan and Tong [3], Lee [4], Nummelin [5], Petrucceli and Woolford [7], Tjosthein [9], Tong [10].

2. Geometric ergodicity

We first consider the following two assumptions:

AI: h is continuous and $q(\cdot)$ is positive everywhere.

AII: h is bounded on bounded sets and $q(\cdot)$ is lower semi-continuous and positive everywhere. It is known that if either AI or AII is satisfied, then $\{X_n\}$ obtained by (1) is irreducible, aperiodic and non-null relatively compact sets are small (see Chan and Tong[2]).

It is desirable to have practical criteria guaranteeing a given Markov chain to be geometrically ergodic or transience, and therefore the aim of this section is to give conditions on h under which the relations in Theorem 1.1 or in Theorem 1.2 are satisfied for suitable test function g.

The basic idea is that (geometric) ergodicity of an irreducible chain may be established by identifying moving back to the centre after a number of steps which depends on the initial position.

For $x = (x^{(1)}, x^{(2)}, \cdots, x^{(k)})' \in R^k$, we define $\|x\|$ as the usual Euclidean norm of x, and $\|x\|_M = \max\{|x^{(i)}|\}$.

THEOREM 2.1. *Suppose that either AI or AII is satisfied with* $E|e_n'| < \infty$ *and that if* $\|x\|$ *is sufficiently large, then*

$$\|h(x)\| \le \beta\|x\|_M + C$$

for some constants β *and* C. *If* $\beta < 1$, *then* $\{X_n\}$ *is geometrically ergodic.*

PROOF. Since $\beta < 1$, we may choose $P_1 > P_2 > \cdots > P_k > 0$ and $0 < \theta < 1$ such that

$$\frac{P_{i+1}}{P_i} < \theta \text{ for } i = 1, 2, \cdots, k-1 \text{ and } \frac{P_1\beta}{P_i} < \theta \text{ for } i = 1, 2, \cdots, k.$$

Now for fixed $P_1, P_2, \cdots, P_k$, define a nonnegative measurable function $g : R^k \to R$ by

$$g(x) = \max_i \left\{ P_i|x^{(i)}| \right\}.$$

If α is sufficiently large, then we have for $\|x\| > \alpha$,

$$\begin{aligned} &E[g(X_{n+1})|X_n = x] \\ &= E\left[\max\{P_1|h(x) + e_{n+1}'|,\ P_2|x^{(1)}|, \cdots, P_2|x^{(k-1)}|\}\right] \\ &\le \max\{P_1\beta\|x\|_M,\ P_2|x^{(1)}|, \cdots,\ P_k|x^{(k-1)}|\} + P_1|C| + P_1E|e_{n+1}'| \\ &\le \theta g(x) + K, \end{aligned}$$

where $K = P_1|C| + P_1E\|e_{n+1}'| < \infty$. By definition of $g(x)$, there exist positive real numbers a and b such that for all $x \in R^k$,

$$a\|x\| \le g(x) \le b\|x\|.$$

Now for $\varepsilon > 0$ if we take $\alpha > \frac{2(K+\varepsilon)}{a(1-\theta)}$ and $B = \{x \in R^k : \|x\| \le \alpha\}$, then

$$E[g(X_{n+1})|X_n = x] \le \left(\frac{1+\theta}{2}\right)g(x) - \varepsilon, \quad \text{for} \quad x \in B^c.$$

Further,

$$E[g(X_{n+1})I_{\{X_{n+1}\in B^c\}}|X_n = x] \le M, \quad x \in B$$

holds for some $M < \infty$ by assumption on h and by definition of g and B.

Hence by Theorem 1.1, the conclusion follows.

COROLLARY 2.2. *Let either AI or AII holds. Suppose that $\alpha > 0$ is sufficiently large and that for $\|x\| > \alpha$,*

$$|h(x)| \leq \sum_{i=1}^{k} a_i |x^{(i)}| \ + \ C$$

where $a_1, a_2, \cdots, a_k$ and C are positive constatns. If $\sum_{i=1}^{k} a_i < 1$, then $\{X_n\}$ is geometrically ergodic.

PROOF. For $\|x\| > \alpha$,

$$|h(x)| \leq \sum a_i |x^{(i)}| \ + \ C \leq \left(\sum a_i\right) \cdot \|x\|_M \ + \ C.$$

Since $\sum a_i < 1, \{X_n\}$ is geometric ergodic by Theorem 2.1.

Above corollary shows that the result in Theorem 2.3 in [5] is a special case of theorem 2.1, and the proof is much simpler.

EXAMPLE. Consider the separable nonlinear autoregressive model, which includes many commenly used models,

$$Z_n \ = \ \alpha_1 h_1(Z_{n-1};\theta_1) + \alpha_2 h_2(Z_{n-2};\theta_2) + \cdots + \alpha_k h_k(Z_{n-k};\theta_k) + e'_n,$$

where $\{e'_n\}$ is a sequence of i. i. d. one-dimensional random variables whose probability density function $q(\cdot)$ is positive everywhere.

Let $\forall \theta_i, \ h_i(\ ;\theta_i)$ be a fixed function bounded over bounded subsets of R.

If we let $X_n = (Z_n, \ Z_{n-1}, \cdots, Z_{n-k+1})$, then $\{X_n\}$ is of the form $X_{n+1} \ = \ f(X_n) \ + \ e_{n+1}$ given by (1). Here $h(x) \ = \ \sum_{i=1}^{k} \alpha_i h_i(x^{(i)};\theta_i)$.

If $|h(x)| \leq \beta \|x\|_M \ + \ C$ for some constants $0 < \beta < 1$ and C, then by Theorem 2.1, $\{X_n\}$ is geometrically ergodic and hence so is $\{Zn\}$ which is not Markovian. If in addition, $h(x)$ satisfies that

(2) $\forall i$, there exists $\phi_i(\alpha_i;\theta_i)$ such that $\alpha_i h_i(x;\theta_i) -\!- \phi_i x$ is bounded,

then the condition that all the roots of $x^k - \phi_1 x^{k-1} - \cdots \phi_k \ = 0$ lie inside the unit circle , which includes the wider region then that of theorem 2.1, guarantees the geometric ergodicity of $\{X_n\}$. But theorem 2.1 deals with the case for which (2) is not satisfed.

3. Transience

Most of the earlier work has been limited to finding sufficient condition for positive recurrence and geometric ergodicity. In this section, we find the condition under which X_n is transient, thus approaching the proof of finding necessary and sufficient condition.

For following two theorems, we assume that either AI or AII is satisfied, so that $\{X_n\}$ generated by (1) is irreducible, aperiodic. Also assume $E\left[e^{|e'_n|}\right] < \infty$. Let μ_k be the Lebesgue measure on R^k.

THEOREM 3.1. *If* $\inf_{\|x\|>\alpha} \frac{|h(x)|+C}{\|x\|_M} > 1$ *for sufficiently large* α *and for some constant* C, *then* $\{X_n\}$ *is transient.*

PROOF. We let a test function $g(x) = e^{-\|x\|_M}$. Then for $\|x\| > \alpha$,

$$\begin{aligned}
&E[g(X_{n+1})|X_n = x] \\
&\le E[e^{-|h(x)+e'_{n+1}|}|X_n = x] \le E[e^{-|h(x)|+|e'_{n+1}|}] \\
&= e^{-|h(x)|}E[e^{|e'_{n+1}|}] \le e^{-(1+\theta)\|x\|_M + K}
\end{aligned}$$

for some $\theta > 0$ and $K = C + ln\ E[e^{|e'_{n+1}|}]$. Now take $B_r^c = \{x : \|x\|_M > r\}$.
We may choose $r > \frac{k}{\theta}$ sufficiently large so that
(1) $\{x : \|x\| > \alpha\} \supset B_r^c$
(2) $\mu_k(B) > 0$ and $\mu_k(B^c) > 0$.
Then for $x \in B_r^c$, $E[g(X_{n+1})|X_n = x] \le g(x)$.
Moreover,

$$e^{-\|x\|_M} \le \inf_{y \in B_r} e^{-\|y\|_M}, \quad x \in B_r^c.$$

Hence by theorem 1.2, $\{X_n\}$ is transient.

THEOREM 3.2. *Let* $|h(x)| \ge \sum a_i|x^{(i)}| + C$ *for* x *with* $\|x\| > \alpha$, α *is sufficiently large. If* $\sum a_i > 1$, *then* $\{X_n\}$ *is transient.*

PROOF. Since $\sum a_i > 1$, we may take $P_k > P_{k-1} > \cdots > P_1 > 0$ with $P_1 < 1$ and $\theta > 1$ such that $\frac{P_{i+1}}{P_i} > \theta$ for $i = 1, 2, \cdots, k-1$ and $\sum_{i=1}^k a_i \frac{P_1}{P_i} > \theta$.

Now we take a test function $g(x)$ as $g(x) = e^{-\min\{P_i|x^{(i)}|\}}$ for fixed $\{P_i\}$ which satisfy the above relation.

Then

$$
\begin{aligned}
& E[g(X_{n+1})|X_n = x] \\
& \le E\left[e^{-\min\{P_1|h(x)+e_{n+1}|, P_2|x^{(1)}|, \cdots, P_k|x^{(k-1)}|\}}\right] \\
(3) \qquad & \le E\left[e^{\max\{-P_1|h(x)|,\ -P_2|x^{(1)}|, \cdots, -P_k|x^{(k-1)}|\} + P_1|e'_{n+1}|}\right] \\
& \le e^{-\theta \min\{P_i|x^{(i)}|\} + |c|} \cdot E\left[e^{P_1|e'_{n+1}|}\right], \quad \text{for } \|x\| > \alpha \\
& \le g(x), \quad \text{for } x \in B^c,
\end{aligned}
$$

where $B^c = \{x : P_i|x^{(i)}| > r, \forall i\}$. Here we choose r sufficiently large so that $r > |C| + \frac{\ln E\left[e^{P_1|e'_{n+1}|}\right]}{\theta - 1}$ and $B^c \subset \{x : \|x\| > \alpha\}$. Hence the relation

$$e^{-\min\{P_i|x^{(i)}|\}} \le \inf_{y \in B} e^{-\min\{P_i|y^{(i)}|\}}, \quad x \in B^c,$$

together with (3) proves that $\{X_n\}$ is transient.

Suppose there exist a fixed integer i and some constant λ and C such that $|h(x)| > \lambda|x^{(i)}| + C$ for x such that $|x^{(i)}|$ is sufficiently large. Then this is a special case of the Theorem 3.2 and hence $\{X_n\}$ is transient.

In this case transience of $\{X_n\}$ can be shown directly by taking $g(x) = e^{-|x^{(i)}|}$ and showing $E[g(X_{n+i})|Xn = x] \le g(x)$, $x \in B^c$, $B = \{x : |x^{(i)}| < r\}$ for sufficiently large r. Since geometric ergodicity of $\{X_{ni}\}$ implies the geometric ergodicity of $\{X_n\}$, the conclusion follows.

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Department of Statistics
Ewha Womans University
Seoul 120-750, Korea