Comm. Korean Math. Soc. 10 (1995), No. 2, pp. 401–407

ON BEST CONSTANTS IN SOME WEAK-TYPE INEQUALITIES

JINSIK MOK

ABSTRACT. The best constants for two distinct weak-type inequalities for martingales and their differential subordinates with values in some spaces isomorphic to a Hilbert space are shown to be the same. This extends the result of Burkholder shown in the Hilbert space setting.

1. Introduction

The aim of this paper is to consider the best constants for some weak-type inequalities for Banach space valued martingales and their differential subordinates and compare them.

Suppose that $\mathbf{X}$ is a real or complex Banach space with norm $|\cdot|$. Let (Ω, F_∞, P) be a probability space and $F = (F_n)_{n\geq 0}$ a nondecreasing sequence of sub-σ-algebras of the σ-algebra F_∞. Then a sequence $f = (f_n)_{n\geq 0}$ of $\mathbf{X}$-valued functions is a *martingale* with respect to the filtration F if each $f_n : \Omega \to X$ is strongly measurable with respect to F_n (the pointwise limit of a sequence of simple F_n-measurable functions) with $\|f_n\|_1 = E|f_n|$ finite and $E(f_{n+1}|F_n) = f_n$. Suppose that f and g are $\mathbf{X}$-valued martingales with respect to the same filtration. Let d be the difference sequence of f and e the difference sequence of g: For all $n \geq 0$, $f_n = \sum_{k=0}^n d_k$ and $g_n = \sum_{k=0}^n e_k$. Then g is *differentially subordinate to* f if, for all $\omega \in \Omega$ and $k \geq 0$,

$$|e_k(\omega)| \leq |d_k(\omega)|.$$

Received July 4, 1994. Revised January 13, 1995.
AMS Classification: 60G42.
Key words: martingales differential subordination. best constant.

Burkholder proved that if $\mathbf{X}$ is a Hilbert space, and if f and g are $\mathbf{X}$-valued martingales with respect to the same filtration and with g differentially subordinate to f, then

$$\lambda P(\sup_{n\geq 0}(|f_n|+|g_n|)\geq\lambda)\leq 2\,\|f\|_1,\quad \lambda>0, \tag{1}$$

and

$$\lambda P(g^*\geq\lambda)\leq 2\,\|f\|_1,\quad \lambda>0, \tag{2}$$

where $\|f\|_1=\sup_{n\geq 0}\|f_n\|_1$ and $g^*(\omega)=\sup_{n\geq 0}|g_n(\omega)|$. The constant 2 is best possible for both inequalities. Futhermore if the best constant in (1) or (2) is 2, then $\mathbf{X}$ is a Hilbert space. See [1, 2, 3, 4].

It is easy to see that the inequality (1) implies the inequality (2). In the next section, we will present a technique by which the converse implication can be shown. Using this technique, we will extend the result of Burkholder to the general Banach space setting.

2. Main results

In this section, $\mathcal{M}$ denotes the set of all pairs (f,g) of $\mathbf{X}$ -valued martingales with respect to the same filtration and with g differentially subordinate to f.

THEOREM 1. *Suppose that there exist a pair (f,g) in $\mathcal{M}$, a nonnegative integer n, and a positive number β so that*

$$P(|f_n|+|g_n|\geq 1)>\beta\,E|f_n|, \tag{3}$$

then there exists a pair (f',g') in $\mathcal{M}$ so that

$$P(|g'_{n+1}|\geq 1)>\beta\,E|f'_{n+1}|. \tag{4}$$

Using Theorem 1, we can extend the result of Burkholder to the general setting:

THEOREM 2. *Let β_1 be the least constant β so that*

$$\lambda P\left(\sup_{n\geq 0}\{|f_n|+|g_n|\}\geq\lambda\right)\leq\beta\,\|f\|_1,\quad (f,g)\in\mathcal{M},\quad \lambda>0. \tag{5}$$

Let β_2 be the least constant β so that

$$\lambda P(g^*\geq\lambda)\leq\beta\,\|f\|_1,\quad (f,g)\in\mathcal{M},\quad \lambda>0. \tag{6}$$

Then $\beta_1=\beta_2$.

Burkholder proved that the best constant in (6) is finite if and only if $\mathbf{X}$ is isomorphic to a Hilbert space. So Theorem 2 makes sense only for Banach spaces isomorphic to a Hilbert space.

3. Proofs of theorems

PROOF OF THEOREM 1. Let (Ω, F_∞, P) be the underlying probability space with the σ-field F_∞ nonatomic. Let $F=(F_n)_{n\geq 0}$ be the smallest filtration to which f and g are adapted, that is, F_n is the smallest σ-field generated by $f_0,\cdots,f_n$ and $g_0,\cdots,g_n$. Denote by d the difference sequence of f and by e the difference sequence of g. For any positive real number α, there is a set $A\in F_\infty$ independent of F_n with $P(A)=\frac{\alpha}{1+\alpha}$. (We can assume that F_∞ is rich enough to allow such an A.)

Let d' be the sequence defined by

$$\begin{aligned}
d'_k &= d_k, \qquad \text{for } 0\leq k\leq n,\\
d'_{n+1} &= \alpha f_n I_{A^c} - f_n I_A,\\
d'_k &\equiv 0, \qquad \text{for } k\geq n+2,
\end{aligned}$$

where A^c denotes the complement of the set A. Assume that g_n never vanishes (if g_n vanishes somewhere on Ω, then this proof can be modified easily), and let e' be the sequence defined by

$$\begin{aligned}
e'_k &= e_k, \qquad \text{for } 0\leq k\leq n,\\
e'_{n+1} &= \frac{|f_n|}{|g_n|}\,g_n I_A - \alpha\frac{|f_n|}{|g_n|}\,g_n I_{A^c},\\
e'_k &\equiv 0, \qquad \text{for } k\geq n+2.
\end{aligned}$$

Then it is easy to see that d' and e' are martingale difference sequences with respect to the new filtration $F' = (F'_n)_{n\geq 0}$, where $F'_k = F_k$, for $k \leq n$, and F'_k is the smallest σ-field containing F_n and A, for $k \geq n+1$.

Let f' be the martingale with the difference sequence d'. Then

$$f'_{n+1} = (\alpha + 1)\, f_n\, I_{A^c},$$

and

$$E(\,|f'_{n+1}|\,|F_n\,) = |f_n|,$$

since A is independent of F_n. Taking expectations on both sides, we obtain

$$E\,|f'_{n+1}| = E\,|f_n|.$$

Let g' be the martingale with the difference sequence e'. Then g' is differentially subordinate to f',

$$g'_{n+1} = g_n + e'_{n+1},$$

and

$$g'_{n+1}|_A = (|f_n| + |g_n|)\,\frac{g_n}{|g_n|}\,I_A,$$

where $g'_{n+1}|_A$ is the restriction of g'_{n+1} to the set A. Thus

$$|g'_{n+1}| = |f_n| + |g_n| \geq 1 \quad \text{on } A \cap \{|f_n| + |g_n| \geq 1\},$$

that is,

$$\{|f_n| + |g_n| \geq 1\} \cap A \subset \{|g'_{n+1}| \geq 1\}.$$

From the assumption (3) together with the monotone convergence theorem, it follows that, for all large α,

$$P(\{|f_n| + |g_n| \geq 1\} \cap A) > \beta\, E\,|f_n|.$$

Since $P(|g'_{n+1}| \geq 1) \geq P(\{|f_n| + |g_n| \geq 1\} \cap A)$ and $E\,|f'_{n+1}| = E\,|f_n|$, we obtain (4). This completes the proof of Theorem 1.

To prove Theorem 2, we need the following:

LEMMA. *The following are equivalent:*
(i) $P(\sup_{n\geq 0}(|f_n| + |g_n|) \geq 1) \leq \beta \, \|f\|_1$, *for all* $(f,g) \in \mathcal{M}$.
(ii) $P(|f_n| + |g_n| \geq 1) \leq \beta \, E|f_n|$, *for all* $n \geq 0$ *and all* $(f,g) \in \mathcal{M}$.

PROOF. Suppose that (i) holds. Select $(f,g) \in \mathcal{M}$ and fix a nonnegative integer n. Consider the martingales f^n and g^n stopped at the n^{th} step:

$$f^n = (f_0, \cdots, f_{n-1}, f_n, f_n, \cdots),$$

$$g^n = (g_0, \cdots, g_{n-1}, g_n, g_n, \cdots).$$

Then the pair (f^n, g^n) belongs to $\mathcal{M}$, $P(|f_n|+|g_n| \geq 1) \leq P(\sup_{n\geq 0}(|f_n| +|g_n|) \geq 1)$, and $\|f^n\|_1 = E|f_n|$.

Since $(f^n, g^n) \in \mathcal{M}$, we obtain, from the assumption (i), that

$$P(|f_n| + |g_n| \geq 1) \leq \beta E|f_n|.$$

For the converse, suppose that (ii) holds. Pick $(f,g) \in \mathcal{M}$, and let τ be the function from Ω to the set of nonnegative integers defined by

$$\tau = \inf\{n \geq 0 : |f_n| + |g_n| > 1\}.$$

For each $k \geq 0$, the set $\{\tau \geq k\}$ is $F_{(k-1)\vee 0}$-measurable, that is, predictable.

Let f^τ and g^τ be the sequences of functions stopped at the random time τ:

$$f_n^\tau = \sum_{k=0}^{n} I_{\{\tau\geq k\}} d_k,$$

$$g_n^\tau = \sum_{k=0}^{n} I_{\{\tau\geq k\}} e_k,$$

where $d = (d_n)_{n\geq 0}$ and $e = (e_n)_{n\geq 0}$ are martingale difference sequences of f and g, respectively. Then both f^τ and g^τ are martingales, g^τ is differentially subordinate to f^τ and

$$\{\sup_{n\geq 0} (|f_n| + |g_n|) > 1\} = \{\tau < \infty\}.$$

If $\omega \in \{|f_n^\tau| + |g_n^\tau| > 1\}$, then $\tau(\omega) \leq n$. So $f_{n+1}^\tau(\omega) = f_n^\tau(\omega)$, $g_{n+1}^\tau(\omega) = g_n^\tau(\omega)$, and $\omega \in \{|f_{n+1}^\tau| + |g_{n+1}^\tau| > 1\}$. Thus the set $\{|f_n^\tau| + |g_n^\tau| > 1\}$ is nondecreasing as n goes to ∞, and

$$\cup_{n=0}^{\infty}\{|f_n^\tau| + |g_n^\tau| > 1\} = \{\sup_{n\geq 0}(|f_n| + |g_n|) > 1\}.$$

By the monotone convergence theorem,

$$P(|f_n^\tau| + |g_n^\tau| > 1) \text{ converges to } P(\sup_{n\geq 0}(|f_n| + |g_n|) > 1),$$

as n goes to ∞, and

$$P(|f_n^\tau| + |g_n^\tau| > 1) \leq \beta E|f_n^\tau| \leq \beta\|f\|_1$$

by the assumption and Doob's optional stopping theorem [5].

The monotone convergence theorem gives

$$P(\sup_{n\geq 0}(|f_n| + |g_n|) > 1) \leq \beta\,\|f\|_1,$$

and by homogeneity,

$$P(\sup_{n\geq 0}(|f_n| + |g_n|) > 1 - 1/\delta) \leq \frac{\beta}{1 - 1/\delta}\|f\|_1, \quad \text{for } \delta > 1.$$

Let δ go to ∞ to obtain

$$P(\sup_{n\geq 0}(|f_n| + |g_n|) \geq 1) \leq \beta\,\|f\|_1.$$

This completes the proof of Lemma.

Proof of Theorem 2. Since $\{g^* \geq \lambda\} \subseteq \{\sup_{n\geq 0}(|f_n|+|g_n|) \geq \lambda\}$, we obtain $\beta_1 \geq \beta_2$. It remains to show that $\beta_2 \geq \beta_1$. By homogeneity, we can take $\lambda = 1$. Take $\beta < \beta_1$. Then there exists a pair $(f^1, g^1) \in \mathcal{M}$ so that

$$P\left(\sup_{n\geq 0}(|f_n^1| + |g_n^1|) \geq 1\right) > \beta\,\|f^1\|_1.$$

By Lemma, there exist a pair $(f^2, g^2) \in \mathcal{M}$ and a nonnegative integer n so that

$$P\left(\,|f_n^2| + |g_n^2| \geq 1\right) > \beta\, E|f_n^2|.$$

By Theorem 1, there exists a pair (f^3, g^3) in $\mathcal{M}$ so that

$$P(|g^3_{n+1}| \geq 1) > \beta\, E|f^3_{n+1}|.$$

By using the stopping time argument as in Lemma, we can show that there exists a pair $(f, g) \in \mathcal{M}$ so that

$$P(g^* \geq 1) > \beta\, \|f\|_1,$$

This shows that $\beta < \beta_2$ and so $\beta_1 \leq \beta_2$, completing the proof of Theorem 2.

References

1. D. L. Burkholder, *Martingale transforms and the geometry of Banach spaces*, Proceedings of the Third International Conference on Probability in Banach spaces,Tufts University, 1980, Lecture Notes in Mathematics **860** (1981), 35-50.
2. D. L. Burkholder, *Sharp inequalities for martingales and stochastic integrals*, Colloque Paul Lévy, Palaiseau, 1987,Astérisque **157-158** (1988), 75-94.
3. D. L. Burkholder, *Differential subordination of harmonic functions and martingales*, Harmonic Analysis and Partial Differential Equations (El Escorial, 1987), Lecture Notes in Mathematics **1384** (1989), 1-23.
4. D. L. Burkholder, *Explorations in Martingale Theory and its Applications*, Saint-Flour Lectures, 1989, Lecture Notes in Mathematics **1464** (1991), 1-66.
5. Doob, *Stochastic Processes*, Wiley, New York, 1953.

Department of Mathematics
Sun Moon University
Choongnam 337-840, Korea