Comm. Korean Math. Soc. **10** (1995), No. 2, pp. 385–392

THE INVARIANT NATURE OF CAUSALITY CONDITIONS ON SPACE-TIMES

JONG CHUL PARK

ABSTRACT. Along with certain properties of a chronal or a causal isomorphism, we discuss the invariant nature of a causality conditions of a given space-time.

1. Introduction

The purpose of this paper is to study the topological and causal structure of a chronally isomorphic or a causal isomorphic image of a given space-time. Vyas and Akolia[11] examined the invariant nature of various causality conditions by using the chronal isomorphism which is definef by Malament[7](however, he calls it causal isomorphism). We shall consider important causal structures, namely, totally vicious, stably causal, and causally simple, which were not discussed in Vyas and Akolia[11]. We show that these can be preserved under a chronal isomorphism or a causal isomorphism.

2. Preliminaries

By a space-time we mean a pair (M, g) with M an orientable, time orientable, connected paracompact and Hausdroff differentiable manifold without boundary and g a Lorentzian metric defined globally on M. The chronological relation $\ll$ (causal relation $\leq$) between points of M are defined by saying $x \ll y$ (causal relation $x \leq y$) if only there is a future-directed timelike (nonspacelike) curve from x to y. The chronological

Received November 10, 1994. Revised December 19, 1994.

1991 AMS Subject Classification: 58.

Key words: chronal isomorphism, causal isomorphism, totally vicious, stably causal, causally simple, Seifert future set.

future $I^+(x)$ and causal future $J^+(x)$ of a point x in M are defined as under. The definitions of $I^-(x)$ and $J^-(x)$ are dual:

$$I^+(x) = \{y \in M : x \ll y\}, J^+(x) = \{y \in M : x \leq y\}$$

We say that p almost causally precedes q, denoted by pAq, if for all $x \in I^-(p), I^+(q) \subset I^+(x)$ or equivalently, if for all $y \in I^+(q), I^-(p) \subset I^-(y)$. We note that pAq iff every neighborhood of p would contain a point which will precede chronologically some points in any neighborhood of q. The almost future $A^+(p)$ and the almost past $A^-(p)$ of $p \in M$ are defined

$$A^+(p) = \{y \in M : pAy\}, A^-(p) = \{x \in M : xAp\}.$$

Seifert[10] defined the $J_s^+(x)$ as follows:

$$J_s^+(x) = \bigcap_{\overline{g}>g} J^+(x, \overline{g})$$

Here $\overline{g} > g$ means that the null corn of $\overline{g}$ are everywhere wider than g, i.e., every nonspacelike vector with respect to g becomes timelike with respect to $\overline{g}$. We call the set $J_s^+(x)$ as the Seifert future set of x.

All of the other notations and terminologies of this paper will be refered to Beem and Ehrlich[1] and Hawking and Ellis[3].

A space-time (M, g) is said to be chronological if $p \notin I^+(p)$ for all $p \in M$. This means that (M, g) contains no closed timelike curves. A space-time (M, g) is said to be causal if there is no pair of distincts points $p, q \in M$ with $p \leq q \leq p$. This is equivalent to requiring (M, g) does not admit any closed nonspacelike curve. Distinguishing conditions were introduced to exclude situations in which there were nonspacelike curves which returned arbitrarily close to their point of origin. A space-time M is distinguishing if $I^+(x) = I^+(y)$ implies $x = y$ and $I^-(x) = I^-(y)$ implies $x = y$. M is said to be strongly causal if for every p in M and any neighborhood N_p of p there is a neighborhood U_p of p contained in N_p, such that no nonspacelike curve intersects U_p more than once. We shall that a space-time (M, g) is stably causal if there exists a Lorentzian metric $\overline{g}$ such such that $(M, \overline{g})$ is causal and for each $p \in M$

and $v \in T_p(M)$ with $v \neq 0$, $g(v,v) \leq 0$ implies $\overline{g}(v,v) < 0$. Further, M is future reflecting if $I^-(x) \subset I^-(y)$ implies $I^+(y) \subset I^+(x)$, and it is past reflecting if $I^+(x) \subset I^+(y)$ implies $I^-(y) \subset I^-(x)$: whereas M is reflecting if it is both past and future reflecting. We say that M is causally continuous if it is distinguishing and reflecting. A distinguishing space-time M is causally simple if $J^+(p)$ and $J^-(p)$ are closed for all $p \in M$. A strongly causal space-time (M,g) is globally hyperbolic, if for each pair of points $p,q \in M$, the set $J^+(p) \cap J^-(q)$ is compact.

3. The invariant nature of causality conditions on space-times

DEFINITION 1. Let (M,g) and (M',g') be two space-times, and let $\phi : M \longrightarrow M'$ be a bijective map.

(a) ϕ is a chronal isomorphism:$x \ll y$ iff $\phi(x) \ll \phi(y)$,

(b) ϕ is a causal isomorphism:$x \leq y$ iff $\phi(x) \leq \phi(y)$.

We note that the definition of chronal isomorphism is same as that Malament's causal isomorphism, but, since it preserves chronological precedence only, we call it chronal isomorphism. Further, it is different from that of Budic and Sachs[2], who also assume that $\phi(x) \leq \phi(y)$, we shall this condition causal isomorphism.

The proofs of following lemmas and propositions are clear from the definition of chronal isomorphism. In the following lemma it is shown that ϕ and I^+ (respectively, I^-) commute.

LEMMA 2. *Let (M,g) and (M',g') be two space-times, and let $\phi : M \longrightarrow M'$ be a chronal isomorphism. Then $\phi(I^+(x))= I^+(\phi(x))$ for all $x \in M$.*

PROPOSITION 3. *Let $\phi : M \longrightarrow M'$ be a chronal isomorphism. Then (M,g) is chronological if and only if (M',g') is chronological.*

PROPOSITION 4. *Let $\phi : M \longrightarrow M'$ be a chronal isomorphism. Then (M,g) is distinguishing if and only if (M',g') is distinguishing.*

PROPOSITION 5. *Let $\phi : M \longrightarrow M'$ be a chronal isomorphism. Then (M,g) is strongly causal if and only if (M',g') is strongly causal.*

PROPOSITION 6. *Let $\phi : M \longrightarrow M'$ be a chronal isomorphism. Then (M, g) is reflecting if and only if (M', g') is reflecting.*

We recall that a space-time M is said to be Woodhouse causal if pAq, and qAp implies $p = q$.

PROPOSITION 7. *Let $\phi : M \longrightarrow M'$ be a chronal isomorphism. Then (M, g) is Woodhouse causal if and only if (M', g') is Woodhouse causal.*

PROOF. By the definition of xAy, we have for all $z \in I^-(x)$, $I^+(y) \subset I^+(z)$. So, applying ϕ we get for all $\phi(z) \in \phi(I^-(x))$, $\phi(I^+(y)) \subset \phi(I^+(z))$. Then the application of Lemma 2 gives, for all $\phi(z) \in I^-(\phi(x))$, $I^+(\phi(y)) \subset I^+(\phi(z))$. Thus we have $\phi(x)A\phi(y)$ implies xAy. Thus xAy if and only if $\phi(x)A\phi(y)$ for all $x, y \in M$. It follows from this fact that chronal isomorphic image of Woodhouse causal space is also Woodhouse causal.

From the Propositions 4, 6, we have the following:

PROPOSITION 8. *Let $\phi : M \longrightarrow M'$ be a chronal isomorphism. Then (M, g) is causally continuous if and only if (M', g') is causally continuous.*

PROPOSITION 9. *Let $\phi : M \longrightarrow M'$ be a chronal isomorphism. Then (M, g) is globally hyperbolic if and only if (M', g') is globally hyperbolic.*

In the following lemma it is shown that ϕ and J^+ (respectively, J^-) also commute under chronal isomorphism.

THEOREM 10. *Let (M, g) and (M', g') be two space-times, and let $\phi : M \longrightarrow M'$ be a chronal isomorphism. Then $\phi(J^+(p)) = J^+(\phi(p))$ for all $p \in M$.*

PROOF. Now if $z = \phi(x) \in \phi(J^+(p))$. Then there a future directed nonspacelike curve from p to x. This curve is timelike or null geodesic. If it is timelike, then $p \ll x$. Since ϕ be chronal isomorphism, $\phi(p) \ll \phi(x)$ if and only if $\phi(x) \in I^+(\phi(p)) \subset J^+(\phi(p))$. Thus $z = \phi(x) \in J^+(\phi(p))$. And if it is a null geodesic, then also there is a null geodesic from $\phi(p)$ to $\phi(x)$ by a result of Hawking, King and MaCarthy[4]. This proves $\phi(J^+(p)) \subset J^+(\phi(p))$. The rest of the part is straightforward, because of the symmetry properties of ϕ and ϕ^{-1}.

The following proposition follows from [6],[11].

PROPOSITION 11. *Let (M, g) and (M', g') be two space-times. Let one of them be distinguishing and $\phi : M \longrightarrow M'$ be a chronal isomorphism. Then ϕ is a homeomorphism. (By Hawking' theorem[4], ϕ is a smooth conformal isometry).*

We get the following from the result of Proposition 11.

THEOREM 12. *Let $\phi : M \longrightarrow M'$ be a chronal isomorphism. Then (M, g) is a causally simple if and only if (M', g') is a causally simple.*

PROOF. Let $\phi(p) = q' \in M'$. Since, by Theorem 10, $J^+(q') = J^+(\phi(p)) = \phi(J^+(p))$, $J^+(q')$ is closed. The rest of the part is straightforward from Proposition 4, 11, thus the proposition is established.

A space-time M is called totally vicious if $I^+(p) \cap I^-(p) = M$ for some (hence all) p in M. There are many equivalent descriptions of this important causality conditions[6],[8].

PROPOSITION 13. *The following are equivalent:*

(a) *M ia totally vicious,*

(b) *The Alexandrov topology on M is trivial, $I^+(p) \cap I^-(q) = M$ for all p, q in M with $p \ll q$.*

THEOREM 14. *Let $\phi : M \longrightarrow M'$ be a chronal isomorphism. Then (M, g) is a totally vicious if and only if (M', g') is a totally vicious.*

PROOF. Let $p', q' \in M'$ with $p' \ll q'$. Then there is $p \ll q$ in M such that $\phi(p) = p', \phi(q) = q'$. Since M is totally vicious, $I^+(p) \cap I^-(q) = M$. Thus, by Lemma 2, $M' = \phi(M) = \phi(I^+(p) \cap I^-(q)) = \phi(I^+(p)) \cap \phi(I^-(q)) = I^+(\phi(p)) \cap I^-(\phi(q)) = I^+(p') \cap I^-(q')$. Therefore M' is totally vicious. Conversely, let $p, q \in M$ with $p \ll q$. Put $\phi(p) = p', \phi(q) = q'$. Since ϕ is is a chronal isomorphism, $p \ll q$ iff $\phi(p) \ll \phi(q)$. So $p' \ll q'$. Thus, $\phi(M) = M' = I^+(p') \cap I^-(q') = I^+(\phi(p)) \cap I^-(\phi(q)) = \phi(I^+(p)) \cap \phi(I^-(q)) = \phi(I^+(p)) \cap (I^-(q))$ Hence $M' = I^+(p) \cap I^-(q)$, so M is totally vicious.

In the following theorem it is shown that ϕ and J^+ (respectively, J^-) commute under causal isomorphism.

Theorem 15. *Let (M, g) and (M', g') be two space-times, and let $\phi : M \longrightarrow M'$ be a causal isomorphism. Then $\phi(J^+(x)) = J^+(\phi(x))$ for all $x \in M$. A similar result holds for J^-.*

Proof. Let $y \in \phi(x)$. There is a point z in $J^+(x)$ with$\phi(z) = y$. Hence, $z \geq x$ and $\phi(z) = y$. Since ϕ be causal isomorphism, $\phi(z) \geq \phi(x)$. Thus $y \in J^+(\phi(x))$. Thus, $\phi(J^+(x)) \subset J^+(\phi(x))$. Now let $s \in J^+(\phi(x))$. Then $s \geq \phi(x)$. So there is a point t in M with $\phi(t) = s$ and $\phi(t) \geq \phi(x)$. Since ϕ is causal isomorphism, $t \geq x$, $\phi(t) = s$. Thus, $\phi(t) \in \phi(J^+(x))$, $\phi(t) = s$. Therefore $s \in \phi(J^+(x))$. It follows that $J^+(\phi(x)) \subset \phi(J^+(x))$. It follows that $J^+(\phi(x)) \subset \phi(J^+(x))$.

Lemma 16. *Let M be a space-time. Then $J_s^+(x)$ is closed for all $x \in M$.*

Lemma 17. *If M is stably causal, then $J_s^+(x) = J_s^+(y)$ implies $x = y$, for all $x, y \in M$.*

Proof. M is stably causal implies whenever $x \in J_s^+(y)$ and $y \in J_s^+(x)$ than $x = y$. Suppose that $J_s^+(x) = J_s^+(y)$. Then $x \in J_s^+(x)$ implies $x \in J_s^+(y)$ and similarly $y \in J_s^+(y)$ implies $y \in J_s^+(x)$. Hence M is stably causal, $x = y$.

The proof of the following proposition is in [5].

Proposition 18. *The following are equivalent:*

(a) *M is stably causal.*

(b) *$h_s(x) \equiv \mu(J_s^+(x))$ is decreasing along all future directed nonspacelike curves. (μ be any finite measure on M)*

We shall show that the stable causality is preserved under a causal isomorphism from the result of Proposition 18.

Theorem 19. *Let $\phi : M \longrightarrow M'$ be a causal isomorphism. Then (M, g) is stably causal if and only if (M', g') is stably causal.*

Proof. Suppose that (M, g) is stably causal. Let γ be a nonspacelike curve in (M', g') and $x', y' \in \gamma$ such that $x' \leq y'$, and $x' \neq y'$. Then there are points x, y in M with $\phi(x) = x', \phi(y) = y', x \neq y$ and $x \leq y$. Hence $J_s^+(y) \subset J_s^+(x)$; however, $x \neq y$ implies $J_s^+(y) \subsetneq J_s^+(x)$. By

Lemma 16, $J_s^+(x)$ is closed for all $x \in M$ and $Int(J_s^+(y)) \subsetneq Int(J_s^+(x))$. Consider now the set

$$A \equiv Int(J_s^+(x)) - Int(J_s^+(y)).$$

Then $Int(A) \neq \emptyset$. If $Int(A) = \emptyset$, then for any $q \in A$ and any neighborhood U, $U \cap Int(J_s^+(y)) \neq \emptyset$ because of $Int(A) = \emptyset$. Since $U \cap Int(J_s^+(y)) \subset U \cap J_s^+(y) \neq \emptyset$, from the definition of boundary, $q \in Bd(J_s^+(y))$. Since$J_s^+(y)$ is closed, $q \in J_s^+(y)$. This implies $Int(J_s^+(x)) \subset J_s^+(y)$, or taking closure, $\overline{Int(J_s^+(x))} = J_s^+(x) \subset \overline{J_s^+(y)} = J_s^+(y)$. Therefore $J_s^+(x) \subset J_s^+(y)$, i.e., $J_s^+(x) = J_s^+(y)$. Since M is stably causal, this is not possible. Hence $Int(A) \neq \emptyset$. Since ϕ is causal isomorphism, $\phi(x) \leq \phi(y), \phi(x) \neq \phi(y)$. Thus, $J_s^+(\phi(y)) \subset J_s^+(\phi(x))$, and then $\mu(J_s^+(\phi(y))) < \mu(J_s^+(\phi(x)))$. Hence $h_s(y') < h_s(x')$. (i.e., h_s decreases along all nonspacelike curves). Thus, by Proposition 18, M' is stably causal. The converse is similar.

References

1. J. K. Beem and P. E. Ehrlich, *Global Lorentzian Geometry*, Marcel Dekker, New York and Bacel, 1981.
2. R. Budic and R.K. Sachs, *Causal boundaries for general relativistic space-times*, J. Math. Phys. 15 (1974), 1302-1309.
3. S. W. Hawking and G. F. R. Ellis, *The large scale Structure of space-time*, Cambridge Monographs on mathematical physics, Cambridge University Press, Cambridge, 1973.
4. S. W. Hawking, A. R. King and P. J. McCarthy, *A new topology for curved space-time which incorporates the causal, differential, and conformal syryctures*, J. Math. Phys. 17 (1976), 174-181.
5. P. S. Joshi, *Causal functions in general relativity II*, Gen. Rel. Grav. 21 (1989), 1227-1231.
6. J. C. Kim and J. H. Kim, *Totally vicious space-time*, J. Math. Phys. 34 (6) (1993), 2435-2439.
7. D. B. Malament, *The class of continuous timelike curves determine the topology of space-times*, J. Math. Phys. 18 (1977), 1399-1404.
8. Y. Matori, *Totally vicious space-times and reflectivity*, J. Math. Phys. 29 (1988), 823-824.
9. B. O'Neill, *Semi-Riemannian Geometry*, Academic Press, New York, 1983.
10. H. J. Seifert, *The causal boundary of space-times*, Gen. Rel. Grav.3 (1971), 247-259.

11. U. D. Vyas, and G.M. Akolia, *Chronal isomorphism*, Gen. Rel. Grav.16 (1984), 1045-1051.
12. U. D. Vyas, and P. S. Joshi, *Causal functions in general relativity*, Gen. Rel. Grav.15 (1983), 553-565.

Department of Mathematics
Yeung Nam University
Kyungsan 712-749, Korea