Comm. Korean Math. Soc. 10 (1995), No. 2, pp. 357–363

L^2-HARMONIC p-FORMS ON A COMPLETE, NON-COMPACT RIEMANNIAN MANIFOLD WITHOUT BOUNDARY

YANG JAE SHIN AND HYO IL CHOI

ABSTRACT. We study non-existence of L^2-harmonic p-forms on a complete, non-compact Riemannian manifold without boundary as extension of results of K. Yano in the case of compact.

1. Introduction

The results of the study of harmonic and Killing tensor fields on a compact Riemannian manifold without boundary had been listed in Yano's book [14]. In [17], non-existence of L^2-Killing vector fields on a complete Riemannian manifold without boundary was discussed and non-existence of L^2-Killing p-forms on a complete, non-compact Riemannian manifold without boundary was studied by the author ([11]).

The study of L^2-harmonic forms on a complete Riemannian manifold has been done in [5] and [15].

The purpose of the present paper is to investigate the properties and non-existence of L^2-harmonic p-forms on a complete, non-compact Riemannian manifold without boundary.

We shall be in C^∞-category. Latin indices run from 1 to n. The Einstein summation convention will be used.

Received July 27, 1994. Revised March 21, 1995.

AMS Classification: 53C12, 53A50.

Key words: global scalar product, cut-off function, $L^2 - p$ form, harmonic p-form, L^2-harmonic p-form.

2. L^2 p-forms on M

Let M be an orientable Riemannian manifold of dimension n and g (resp.∇) the Riemannian metric (resp. the Riemannian connection) on M.

We consider a p-form on M

$$\omega = \frac{1}{p!}\omega_{i_1\dots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}. \tag{2.1}$$

or a *skew-symmetric tensor field* of type $(0,p)$. Then $\omega_{i_1\dots i_p}$ are local components of the p-form ω. The *exterior differential* $d\omega$ of a p-form ω on M is a $(p+1)$-form given by

$$\begin{aligned} d\omega = & \frac{1}{(p+1)!}\{\nabla_i\omega_{i_1\dots i_p} - \nabla_{i_1}\omega_{ii_2\dots i_p} - \dots \\ & - \nabla_{i_p}\omega_{i_1\dots i_{p-1}i}\} dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p}. \end{aligned} \tag{2.2}$$

From a p-form ω on M, the $(p-1)$-form given by

$$\delta\omega = -\frac{1}{(p-1)!} g^{ji}\nabla_j\omega_{ii_2\dots i_p} dx^{i_2} \wedge \cdots \wedge dx^{i_p} \tag{2.3}$$

is called *the codifferential* of the p-form ω. If ω is a function on M, then we put $\delta\omega = 0$.

The *Laplace-Beltrami operator* $\Delta = \delta d + d\delta$ on $\Lambda^p(M)$ is represented by

$$\begin{aligned} \Delta\omega = & \delta d\omega + d\delta\omega \\ = & -\frac{1}{p!}\{g^{ji}\nabla_j\nabla_i\omega_{i_1\dots i_p} - \sum_{s=1}^{p} K_{i_s}{}^t\omega_{i_1\dots t\dots i_p} \\ & - \sum_{t<s}^{1\dots p} K_{i_t i_s}{}^{ab}\omega_{i_1\dots a\dots b\dots i_p}\} dx^{i_1} \wedge \cdots \wedge dx^{i_p} \end{aligned} \tag{2.4}$$

by the Ricci identity and $K_s{}^t = -g^{ji}K_{jsi}{}^t$ for any p-form ω, where $K_{kji}{}^h$ and K_{ji} are local components of the *Riemannian curvature tensor* and *Ricci tensor* of M, respectively.

From now on we assume that M is a complete, non-compact, connected and orientable Riemannian manifold of dimension n without boundary unless specially mentioned.

Let $\Lambda^P(M)$ be the space of all p-forms on M and $\Lambda_0^P(M)$ the subspace of $\Lambda^P(M)$ composed of forms with compact supports.

The Hodge $*$-operator on $\Lambda^P(M)$ is defined by (cf. [6], [10])

$$(2.5)\qquad \begin{aligned} *\omega = & \sum_{j_1<\cdots<j_p;k_1<\cdots<k_{n-p}} g^{i_1j_1}\cdots g^{i_pj_p}\delta^{1\cdots\cdots\cdots\cdots n}_{j_1\cdots j_p k_1\cdots k_{n-p}} \\ & \times\sqrt{det(g_{ji})}\omega_{i_1\cdots i_p}dx^{k_1}\wedge\cdots\wedge dx^{k_{n-p}}, \end{aligned}$$

where $\omega = \frac{1}{p!}\omega_{i_1\cdots i_p}dx^{i_1}\wedge\cdots\wedge dx^{i_p}$ and $\delta^{1\cdots\cdots\cdots\cdots n}_{j_1\cdots j_p k_1\cdots k_{n-p}}$ denotes the Kronecker symbol. Thus we may define a global scalar product $<<\ ,\ >>$ on $\Lambda_0^P(M)$ by (cf. [4], [9], [10], [16])

$$(2.6)\qquad << \phi,\psi >> = \int_M < \phi,\psi > dV = \int_M \phi\wedge *\psi.$$

Also, we have (cf. [5], [16])

$$(2.7)\qquad << d\phi,\psi >> = << \phi,\delta\psi >>$$

for any $\phi\in\Lambda_0^P(M)$ and $\psi\in\Lambda_0^{P+1}(M)$.

Let x_0 be a fixed point of M and $\rho(p)$ the distance from x_0 to $p\in M$. Then the set

$$(2.8)\qquad B(2\alpha) = \{p\in M|\ \rho(p)\le 2\alpha\}$$

is compact in M for any $\alpha > 0$.

On the other hand, if we consider a cut-off function μ on R satisfying (cf. [9], [15])

$$(2.9)\qquad \begin{cases} 0\le\mu\le 1 & \text{on } R, \\ \mu(y) = 1 & \text{for } y\le 1, \\ \mu(y) = 0 & \text{for } y\ge 2, \end{cases}$$

then we can define a family $\{\lambda_\alpha\}$ of Lipschitz continuous functions on M by (cf. [7], [16])

$$\lambda_\alpha(q) = \mu(\rho(q)/\alpha), \quad \alpha = 1, 2, 3, \ldots \tag{2.10}$$

for any $q \in M$. Thus the family $\{\lambda_\alpha\}$ satisfies the following properties:

$$\begin{cases} 0 \le \lambda_\alpha(q) \le 1 & \text{for any } p \in M, \\ \mathrm{supp}\lambda_\alpha \subset B(2\alpha), & \\ \lambda_\alpha(q) = 1 & \text{for any } q \in B(\alpha), \\ \lim_{\alpha \to \infty} \lambda_\alpha = 1, & \\ \|d\lambda_\alpha\| \le D\alpha^{-1} & \text{almost everywhere on } M, \end{cases} \tag{2.11}$$

where D is a positive constant independent on α (cf. [1], [3], [6], [16], [17]).

In fact, ρ is locally Lipschitz function and $\|d\rho\|^2 \le n$. Since $d\lambda_\alpha = \frac{1}{\alpha}(d\mu/dt)d\rho$ at the point where the derivative of ρ exists (cf. [8]), setting $A := \sup \|\frac{d\mu}{dt}\|$ implies (cf. [2])

LEMMA 2.1. *Under the above notations,*

$$\|d\lambda_\alpha \wedge \omega\|^2_{B(2\alpha)} \le \frac{nA^2}{\alpha^2}\|\omega\|^2_{B(2\alpha)}, \tag{2.12}$$

for any $\omega \in \Lambda^P_o(M)$, where A is a positive constant depending only μ and $\|\omega\|^2_{B(2\alpha)} = << \omega, \omega >>_{B(2\alpha)} = \int_{B(2\alpha)} < \omega, \omega > *1$.

Let $L^2_P(M)$ be the completion of $\Lambda^P_o(M)$ with respect to the global scalar product $<< \, , \, >>$. A tensor field $\omega \in L^2_P(M) \cap \Lambda^P(M)$ is called the L^2-*p-form* on M. Then we remark that $\|\omega\| < \infty$, $\lambda_\alpha\omega \in \Lambda^P_o(M)$ and $\lambda_\alpha\omega \to \omega$ as $\alpha \to \infty$ in the strong sense for any L^2-p-form ω on M (cf. [2], [7], [16]).

3. Non-existence of L^2-harmonic p-forms on M

In this section, we will find useful properties of L^2-harmonic p-form on M. From these properties, we will obtain our main result which is a natural extension of that of K. Yano ([13], [14]) in the case of compact Riemannian manifold. Furthermore, we will study relations between curvature (or Ricci) tensor and non-existence of L^2-harmonic p-forms on M.

A p-form ω on M is a *harmonic p-form* if it satisfies

$$d\omega = 0 \quad \text{and} \quad \delta\omega = 0 \tag{3.1}$$

We first introduce the following lemma due to T. Takahashi ([12]).

LEMMA 3.1. *A p-form ω on M is a harmonic form if and only if*

$$\Delta\omega = 0 \tag{3.2}$$

or

$$g^{ji}\nabla_j\nabla_i\omega_{i_1\ldots i_p} - K_{i_s}{}^t\omega_{i_1\ldots t\ldots i_p} - \sum_{t<s}^{1\ldots p} K_{i_t i_s}{}^{ab}\omega_{i_1\ldots a\ldots b\ldots i_p} = 0 \tag{3.3}$$

We consider (the square length) $|\lambda_\alpha\omega|^2$ of a L^2-p-form ω on M. Then we have

$$\frac{1}{2}\Delta|\lambda_\alpha\omega|^2 = < \delta\nabla(\lambda_\alpha\omega), \lambda_\alpha\omega > \; - \; |\nabla(\lambda_\alpha\omega)|^2. \tag{3.4}$$

Suppose that ω is a L^2-harmonic p-form, then we have

$$(\delta\nabla\omega)_{i_1\cdots i_p} = -\sum_{s=1}^{p} K_{i_s}^t\omega_{i_1\cdots t\cdots i_p} - \sum_{t<s}^{1\cdots p} K_{i_t i_s}^{ab}\omega_{i_1\cdots a\cdots b\cdots i_p}, \tag{3.5}$$

REMARK 3.2. For any harmonic p-form ω on M, if $F_p(\omega,\omega)$ is the quadratic form given by

$$F_p(\omega,\omega) = K_{ji}\omega^j{}_{i_2\ldots i_p}\omega^{ii_2\ldots i_p} + \frac{p-1}{2}K_{kjih}\omega^{kj}{}_{i_3\ldots i_p}\omega^{ihi_3\ldots\ldots i_p}, \tag{3.6}$$

where K_{ji} and K_{kjih} is the covariant components of the Ricci tensor R and the curvature tensor K, respectively, then it follows that

$$< \delta\nabla\omega, \omega > = -pF_p(\omega,\omega). \tag{3.7}$$

On the other hand, since $F_p(\omega,\omega)$ is bilinear, $< \lambda_\alpha\delta\nabla\omega, \lambda_\alpha\omega > = F_p(\lambda_\alpha\omega, \lambda_\alpha\omega)$. Thus we have the following useful formula:

LEMMA 3.3. *If ω is a L^2-harmonic p-form on M, then it holds that*

$$\frac{1}{2}\Delta|\lambda_\alpha\omega|^2 = <\delta d\lambda_\alpha, \lambda_\alpha|\omega|^2> - |d\lambda_\alpha \wedge \omega|^2 - \lambda_\alpha^2|\nabla\omega|^2 - pF_p(\lambda_\alpha\omega, \lambda_\alpha\omega) - 2<\lambda_\alpha\nabla\omega, 2d\lambda_\alpha \wedge \omega>. \tag{3.8}$$

Using Stokes' theorem and Lemma 3.3, we have our main results:

THEOREM 3.4. *Let M be a complete, non-compact, connected and orientable Riemannian manifold of dimension n without boundary. If the quadratic form $F_p(\omega,\omega)$ is positive-semidefinite, then any L^2-harmonic p-form on M is parallel.*

PROOF. By Stokes' theorem, we have

$$\frac{1}{2}\int_{B(2\alpha)} \Delta|\lambda_\alpha\omega|^2 dV = -\frac{1}{2}\int_{\partial B(2\alpha)} <N, d|\lambda_\alpha\omega|^2> dB, \tag{3.9}$$

where N is the outer normal vector to $\partial B(2\alpha)$ and dB is the volume element of $\partial B(2\alpha)$.

Since $\partial B(2\alpha) = \partial M \cup \{p \in M | \rho(p) = 2\alpha\}$, $\lambda_\alpha = 1$ on ∂M and $\lambda_\alpha = 0$ on $\{p \in M | \rho(p) = 2\alpha\}$ (cf. [2]). Moreover, since $\partial M = \phi$, the right hand side of (3.9) is equal to zero. Thus from Lemma 3.3, we have

$$\begin{aligned} 0 &\le |\int_{B(2\alpha)} pF_p(\lambda_\alpha\omega, \lambda_\alpha\omega)dV| + |\int_{B(2\alpha)} \lambda_\alpha^2|\nabla\omega|^2 dV| \\ &= |\int_{B(2\alpha)} |d\lambda_\alpha \wedge \omega|^2 dV| + |\int_{B(2\alpha)} <d\lambda_\alpha, d(\lambda_\alpha|\omega|^2)> dV| \\ &\quad + 2|\int_{B(2\alpha)} <\lambda_\alpha\nabla\omega, 2d\lambda_\alpha \wedge \omega> dV| \\ &\le \|d\lambda_\alpha \wedge \omega\|^2_{B(2\alpha)} + <<d\lambda_\alpha, d(\lambda_\alpha|\omega|^2)>>_{B(2\alpha)} \\ &\quad + 2<<\lambda_\alpha\nabla\omega, 2d\lambda_\alpha \wedge \omega>>_{B(2\alpha)}. \end{aligned}$$

Letting $\alpha \to \infty$, from Lemma 2.1 we have

$$p\int_M F_p(\omega,\omega)dV + \int_M |\nabla\omega|^2 dV = 0.$$

Hence if $F_p(\omega,\omega)$ is positive-semidefinite, then $\nabla\omega = 0$.

COROLLARY 3.5. *If the quadratic form $F_p(\omega,\omega)$ is positive-semidefinite on M and $F_p(\omega,\omega) > 0$ at some point in M, then there are no non-zero L^2-harmonic p-forms on M.*

References

1. A. Andreotti and E. Vesentini, *Carleman estimates for the Laplace-Beltrami equation on complex manifolds*, Inst. Hautes Etudes Sci. Publ. Math. **25** (1965), 313 - 362.
2. T. Aoki, J. S. Pak and S. Yorozu, *L^2-vector fields on a non-compact Riemannian manifold with boundary*, Ann. Sci. Kanazawa Univ. **25** (1988), 17 - 29.
3. T. Aoki and S. Yorozu, *L^2-transverse conformal and Killing fields on complete foliated Riemannian manifolds*, Yokohama Math. J. **36** (1988), 27 - 41.
4. J. Dodziuk, *L^2-harmonic forms on rotationally symmetric Riemannian manifolds*, Proc. Amer. Math. Soc. **77** (1979), 395 - 400.
5. J. Dodziuk, *L^2-harmonic forms on complete manifolds, Seminar on Differential Geometry, ed. S.T. Yau*, Ann. of Math. Studies, No. 102 Princeton Univ. Press, Princeton, N.J. (1982).
6. S. I. Goldberg, *Curvature and Homology*, Academic Press, New York, 1962.
7. H. Kitahara, *Nonexistence of nontrivial $\square''$-harmonic 1-forms on a complete foliated Riemannian manifold*, Trans. Amer. Math. Soc. **262** (1980), 429 - 435.
8. H. Kitahara and H. Matsuda, *non-existence of non-trivial harmonic forms on a complete Riemannian manifold with a boundary*, Ann. Sci. Kanazawa Univ. **20** (1983), 1-12.
9. K. Okamoto and H. Ozeki, *On square-integrable $\overline{\partial}$-cohomology spaces attached to hermitian symmetric spaces*, Osaka J. Math. **4** (1967), 95 - 110.
10. B. L. Reinhart, *Harmonic integrals on foliated manifolds*, Amer. J. Math. **81** (1959), 529 - 536.
11. Y. J. Shin, *L^2-Killing p-forms on a complete, non-compact Riemannian manifold without boundary*, Kyungpook Math. J. **32 No. 1** (1992), 131 - 140.
12. T. Takahashi, *On harmonic and Killing tensor fields in a Riemannian manifold with boundary*, J. Math. Soc. Japan **14** (1962), 37 - 65.
13. K. Yano, *Some remarks on tensor fields and curvature*, Ann. of Math. **55** (1952), 328 - 347.
14. K. Yano, *Integral Formulas in Riemannian Geometry*, Marcel Dekker INC, New York, 1970.
15. S. T. Yau, *Problems section, Seminar on Differential Geometry*, Ann. of Math. Studies, No. 102, Princeton Univ. Press, Princeton N J. (1982).
16. S. Yorozu, *Notes on square-integrable cohomology spaces on certain foliated manifolds*, Trans. Amer. Math. Soc. **255** (1979), 329 - 341.
17. S. Yorozu, *Killing vector fields on complete Riemannian manifolds*, Proc. Amer. Math. Soc. **84** (1982), 115 - 120.

Department of Math. Education
Kyungnam University
Masan 631-701, Korea