Comm. Korean Math. Soc. 10 (1995), No. 2, pp. 349–355

STABILITY OF THE BERGMAN KERNEL FUNCTION ON PSEUDOCONVEX DOMAINS IN $\mathbb{C}^n$

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ABSTRACT. Let $D \Subset \mathbb{C}^n$ be a smoothly bounded pseudoconvex domain and let $\{\overline{D}_\tau\}_\tau$ be a family of smooth perturbations of $\overline{D}$ such that $\overline{D} \subset \overline{D}_\tau$. Let $K_D(z,w)$ be the Bergman kernel function on $D \times D$. Then $\lim_{\tau \to 0} K_{D_\tau}(z,w) = K_D(z,w)$ locally uniformally on $D \times D$.

1. Introduction

In [5], Greene and Krantz got the stability, under C^∞ perturbations of a strongly pseudoconvex domain, of the Bergman kernel function. In this paper, we investigate a stability result for the Bergman kernel function on weakly pseudoconvex domains in $\mathbb{C}^n$. This is a consequence of approximation theorems for square integrable holomorphic functions and extremal properties of the Bergman kernel function.

Let $\overline{M}$ be a compact pseudoconvex complex manifold with C^∞ boundary such that the complex structure of M extends smoothly up to the boundary. Let $M' \supset \overline{M}$ be a smooth manifold. In a local coordinate chart U of a complex manifold $\overline{M}$, any C^∞ vector field can be written as $L = \sum_{j=1}^{2n} a_j \frac{\partial}{\partial x_j}$ where $a_j \in C^\infty(U)$ and $x_j, j = 1, 2, \ldots, 2n$, are local coordinate (in a real sense) functions. Therefore we can define C^∞-metric for a C^∞ vector field L by

$$\|L\|_{C^\infty(U)} = \max_j\{\|a_j\|_{C^\infty(U)}\}$$

where $\|a_j\|_{C^\infty(U)}$ is a C^∞-metric for C^∞ functions.

Received November 10, 1994.

1991 Mathematics Subject Classification: 32H10, 32F20.

Key words and phrases: Stability, perturbation, Bergman kernel function, $\overline{\partial}$-Neumann problem, holomorphic approximation.

This work was supported by Korea Research Foundation, 1994.

DEFINITION. Let $I \subset \mathbb{R}^l$ be a domain containing 0. Then the family of complex manifolds $\{M_\tau\}_{\tau \in I}$, $M_\tau \subset M'$, with smooth defining functions r_τ for $\tau \in I$, is said to be a continuous family of diffeomorphic complex manifolds with diffeomorphisms $d_\tau : M_\tau \to M_0$, if

(1) $d_0 : M_0 \to M_0$ is an identity,
(2) the complex structures on M_τ are C^∞ close to the complex structure on M_0 as $\tau \to 0$,
(3) r_τ and all of its derivatives are continuous functions with respect to τ,
(4) the diffeomorphisms d_τ, defined on M_τ, are continuous functions of τ.

REMARK. If $D \Subset \mathbb{C}^n$ is a smoothly bounded pseudoconvex domain of finite type in the sense of D'Angelo, then we can construct a continuous family $\{\overline{D}_\delta\}_{0 \le \delta \le \delta_0}$ of smoothly bounded diffeomorphic compact pseudoconvex domains in $\mathbb{C}^n$ with $\overline{D} = \overline{D}_0 \subset D_\delta$ for all $0 < \delta \le \delta_0$ (See [4]). Furthermore, such non-trivial continuous families $\{\overline{D}_\delta\}_\delta$ can exist with $D = D_0 \subset D_\delta$ for all δ even if bD is not finite type.

THEOREM. *Let $\{\overline{D}_\tau\}_{\tau \in I}$ be a continuous family of smoothly bounded diffeomorphic compact pseudoconvex domains in $\mathbb{C}^n$ such that $D = D_0 \subset D_\tau$ for all $\tau \in I$. Then*

$$K_D(z,w) = \lim_{\tau \to 0} K_{D_\tau}(z,w)$$

locally uniformly on $D \times D$.

2. Holomorphic approximation theorems

Let $\zeta_j, j = 1, 2, \ldots, N$, be a partition of unity subordinated to a covering of $\overline{M}$. We define the weighted m-th Sobolev norm as follows

$$\|f\|^2_{m,t,M} = \sum_{j=1}^{N} \sum_{|\alpha| \le m} \|D^\alpha(\zeta_j f)\|^2_{t,M}$$

where D^α refers to $\partial^{|\alpha|}/\partial x_1^{\alpha_1} \cdots \partial x_{2n}^{\alpha_{2n}}$ for the coordinate neighborhood of $supp\, \zeta_j$ and $\| \cdot \|_{t,M}$ is the weighted L^2-norm with weight $e^{-t\varphi}$. Here

φ is a smooth function on $\overline{M}_\tau$ for all $\tau \in I$. Let $H^{0,1}(M_\tau)$ be the $\overline{\partial}$-cohomology group on M_τ.

In [2], we got the following stability result of the Catlin estimate [1, Proposition 2.3.2] to the $\overline{\partial}$-Neumann problem on pseudoconvex complex manifolds. By using this result, we will prove the holomorphic approximation theorem on continuous families of diffeomorphic compact pseudoconvex complex manifolds.

LEMMA 2.1. ([3]) *Let $\{\overline{M}_\tau\}_{\tau\in I}$ be a continuous family of smoothly bounded diffeomorphic compact pseudoconvex complex manifolds in M'. Suppose there is a C^∞ plurisubharmonic function φ defined on a neighborhood of $\cup_{\tau\in I}\overline{M}_\tau$ which is strongly plurisubharmonic on a neighborhood of bM_τ for all $\tau \in I$. Let m be a nonnegative integer. Then there exists a constant $C_{m,t}$ which does not depend on τ, and there is a neighborhood I' of $\tau = 0$, such that*

$$\|f^\tau\|_{m,t,M_\tau} \leq C_{m,t}\|\square_\tau f^\tau\|_{m,t,M_\tau}$$

for all $f^\tau \in D_{\square_\tau} \cap C^\infty(\overline{M}_\tau)$ with $f^\tau \perp H^{0,1}(M_\tau)$, $\tau \in I'$. Here $\square_\tau$ denotes the usual complex Laplacian on M_τ and $D_{\square_\tau}$ denotes the domain of $\square_\tau$.

We denote by $A^2(M)$ the set $H(M)\cap L^2(M)$, and by $A^\infty(M)$ the set $H(M)\cap C^\infty(\overline{M})$. Let $H_m(M)$ be the Sobolev space of order m.

LEMMA 2.2. ([1]) *Let $\overline{M}$ be a smoothly bounded compact pseudoconvex complex manifold. Assume that there exists a function $\varphi \in C^\infty(\overline{M})$ such that φ is strongly plurisubharmonic in a neighborhood of the boundary bM. Let f be a function in $H(M)\cap H_m(M)$, where m is a nonnegative integer. Then for any $\epsilon > 0$, there exists $g \in A^\infty(M)$ with $\|g - f\|_{m,M} < \epsilon$.*

THEOREM 2.3. *Let $\{\overline{M}_\tau\}_{\tau\in I}$ and φ be as in Lemma 2.1. Let f be a function in $A^2(M)$. Then there exist $f_\tau \in A^2(M_\tau)$ such that*

$$\lim_{\tau\to 0}\|f - f_\tau\|_M = 0 \quad \text{and} \quad \lim_{\tau\to 0}\|f_\tau\|_{M_\tau} = \|f\|_M.$$

PROOF. By Lemma 2.2, $A^\infty(M)$ is dense in $A^2(M)$. Thus we may assume that $f \in A^\infty(M)$. We define $f_\tau = P_t^\tau(d_\tau^* f)$, where $d_\tau : \overline{M}_\tau \to \overline{M}_0 = \overline{M}$ are diffeomorphisms and $P_t^\tau : L^2(M_\tau, t\varphi) \to L^2(M_\tau, t\varphi) \cap H(M_\tau)$ are Bergman projections with respect to the weight $e^{-t\varphi}$. Then f_τ is holomorphic on M_τ and

$$f_\tau = d_\tau^* f - \overline{\partial}_\tau^* N_t^\tau \overline{\partial}_\tau d_\tau^* f,$$

where N_t^τ are Neumann operators on M_τ with respect to the weight $e^{-t\varphi}$. By Lemma 2.1,

$$\begin{aligned} \|\overline{\partial}_\tau^* N_t^\tau \overline{\partial}_\tau d_\tau^* f\|_{t,M_\tau} &\lesssim \|N_t^\tau \overline{\partial}_\tau d_\tau^* f\|_{1,t,M_\tau} \\ &\lesssim \|\overline{\partial}_\tau d_\tau^* f\|_{1,t,M_\tau} \end{aligned}$$

uniformly in τ near 0. Since the complex structures on $\overline{M}_\tau$ converges to the complex structure on $\overline{M}_0 = \overline{M}$ in C^∞-topology, we can get $\overline{\partial}_\tau d_\tau^* f \to \overline{\partial} f = 0$, also in C^∞-topology as $\tau \to 0$. So $\|f_\tau - d_\tau^* f\|_{t,M_\tau}$ converges to zero as $\tau \to 0$. Since the diffeomorphisms d_τ are continuous functions of τ, $d_\tau^* f \to f$ in C^∞-topology on $\overline{M}_0$ and $|\det \operatorname{Jac}_{\mathbb{C}} d_\tau| \to 1$ uniformly as $\tau \to 0$. Thus there exists a neighborhood I' of $\tau = 0$ such that

$$\|d_\tau^* f - f\|_{t,M} < \frac{\epsilon}{2} \quad \text{and} \quad \big|\|d_\tau^* f\|_{t,M_\tau} - \|f\|_{t,M}\big| < \frac{\epsilon}{2}$$

for each $\tau \in I'$. Since $\varphi \in C^\infty(\cup_{\tau \in I} \overline{M}_\tau)$, the weighted L^2-norms are equivalent to the usual L^2-norms. Thus $f_\tau \in A^2(M_\tau)$ for each $\tau \in I'$ and, by shrinking I' if necessary, we get

$$\|f - f_\tau\|_M < \epsilon \quad \text{and} \quad \big|\|f_\tau\|_{M_\tau} - \|f\|_M\big| < \epsilon.$$

Hence we get the theorem 2.3. □

3. Proof of Theorem

For every $z \in D$ there is an extremal function $f \in A^2(D)$ such that $\|f\|_D = 1$ and $|f(z)|^2 = K_D(z, z)$. By Theorem 2.3, for any $\epsilon > 0$, there exists a neighborhood I' of $\tau = 0$ such that for each $\tau \in I'$ there is $f_\tau \in A^2(D_\tau)$ satisfying

$$\|f - f_\tau\|_D < \epsilon \quad \text{and} \quad \|f_\tau\|_{D_\tau} < 1 + \epsilon. \tag{3.1}$$

For all $z \in D$ one has

$$K_D(z,z) = \sup\{\frac{|f(z)|^2}{\|f\|_D^2} : f \in A^2(D), f \neq 0\}.$$

Thus, from (3.1), it follows that

$$|f(z) - f_\tau(z)|^2 < \epsilon^2 K_D(z,z) \quad \text{and} \quad K_{D_\tau}(z,z) > (1+\epsilon)^{-2}|f_\tau(z)|^2.$$

Hence we get

$$K_{D_\tau}(z,z) \geq (1+\epsilon)^{-2}(1-\epsilon)^2 K_D(z,z).$$

Since $K_{D_\tau}(z,z) \leq K_D(z,z)$, we have

$$K_D(z,z) = \lim_{\tau \to 0} K_{D_\tau}(z,z). \tag{3.2}$$

Let C be a compact subset of D. Then for $z, w \in C$ we have

$$\begin{aligned} |K_{D_\tau}(z,w)| &\leq \sqrt{K_{D_\tau}(z,z)}\sqrt{K_{D_\tau}(w,w)} \\ &\leq \sup\{K_D(z,z) : z \in C\} < \infty. \end{aligned}$$

Hence the sequence $\{K_{D_\tau}\}_{\tau \in I}$ is locally bounded on $D \times D$ which provides a subsequence $\{K_{D_{\tau_j}}\}_j$ with

$$\lim_{j \to \infty} K_{D_{\tau_j}}(z,w) =: k(z,w). \tag{3.3}$$

Here the convergence is locally uniform: so the function $D \times \overline{D} \ni (z,w) \mapsto k(z,\overline{w})$ is holomorphic, where $D \times \overline{D} = \{(z,\overline{w}) | z, w \in D\}$.

Thus it is enough to show that $k(\cdot,w)=K_D(\cdot,w)$. From (3.2) and (3.3) it follows that

$$\begin{aligned}\|k(\cdot,w)\|_D^2 &= \int_D |k(z,w)|^2 dV(z)\\ &= \lim_{j\to\infty}\int_D |K_{D_{r_j}}(z,w)|^2 dV(z)\\ &\le \lim_{j\to\infty}\int_{D_{r_j}} |K_{D_{r_j}}(z,w)|^2 dV(z)\\ &= \lim_{j\to\infty} K_{D_{r_j}}(w,w)\\ &= k(w,w)=K_D(w,w),\ w\in D.\end{aligned}$$

Thus $k(\cdot,w)\in A^2(D)$ for every $w\in D$ and $k(w,w)=0$ implies $k(\cdot,w)=0$. Since $|K_D(z,w)|^2\le K_D(z,z)K_D(w,w)$, $k(w,w)=K(w,w)=0$ implies $K_D(\cdot,w)=0$. Therefore, if $k(w,w)=0$, then we get $k(\cdot,w)=K_D(\cdot,w)=0$. But if $k(w,w)\ne 0$, then

$$\tag{3.4}\begin{aligned}\left\|\frac{k(\cdot,w)}{k(w,w)}\right\|_D &= \frac{1}{\sqrt{k(w,w)}}\frac{\|k(\cdot,w)\|_D}{\sqrt{k(w,w)}}\le\frac{1}{\sqrt{k(w,w)}}\\ &= \frac{1}{\sqrt{K_D(w,w)}}=\left\|\frac{K_D(\cdot,w)}{K_D(w,w)}\right\|_D.\end{aligned}$$

Recall that the standard relationship between the Bergman kernel function and the extremizing functions: For any $w\in D$, there is a unique function $g\in A^2(D)$ such that $g(w)=1$ and $\|g\|_D\le\|f\|_D$ for any $f\in A^2(D)$ with $f(w)=1$. In fact, $g:=\frac{K_D(\cdot,w)}{K_D(w,w)}$ is the only extremizing function in $A^2(D)$. Therefore, by (3.4), it follows that $k(\cdot,w)=K_D(\cdot,w)$. □

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