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SOME FIXED POINT THEOREMS ON H-SPACES(I)

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ABSTRACT. In this paper we obtain some fixed point theorems on H-spaces by using H-KKM theorems.

1. Introduction and Preliminaries

The famous Knaster-Kuratowski-Mazurkiewicz theorem[6] (for short, KKM theorem) has been extended to the infinite dimensional spaces by many authors[1, 2, 4, 7, 9]. It has been well-known that several kinds of KKM theorems play important roles in nonlinear analysis. Recently Horvath[4] generalized KKM theorems, by replacing convexity assumptions with merely topological property, i.e., contractibility. Since then, Bardaro-Ceppitelli[1], Tarafdar[9], Chang[2], and Park[7] have considered generalized KKM theorem which are versions of the classical KKM theorems to Horvath spaces (shortly, H-spaces), and obtained many important results by using the KKM-techniques in H-spaces.

In this paper, we establish some fixed point theorems on H-spaces by using H-KKM theorems, which are generalizations of Park's results[8] obtained in convex spaces.

Now we give some definitions and results needed in section 2 .

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DEFINITION 1.1. [1] An H-space is a pair $(X, \{\Gamma_A\})$, where X is a topological space, and $\{\Gamma_A\}$ is a given family of nonempty contractible subsets of X, indexed by the finite subsets of X such that $A \subset B$ implies $\Gamma_A \subset \Gamma_B$. Let $(X, \{\Gamma_A\})$ be an H-space, a subset $D \subset X$ is said to be H-convex if for every finite subset, it follows that $\Gamma_A \subset D$. A subset $D \subset X$ is said to be weakly H-convex if, for every finite subset $A \subset D$, it results that $\Gamma_A \bigcap D$ is nonempty and contractible. Finally, a subset $K \subset X$ is said to be H-compact if , for every finite subst $A \subset X$, there exists a compact weakly H-covex set $D \subset X$ such that $K \bigcup A \subset D$.

REMARK. If a subset $D \subset X$ is weakly H-covex, then it is equivalent to saying that the pair $(D, \{\Gamma_A \bigcap D\})$ is an H-space.

DEFINITION 1.2. [9] Let $(X, \{\Gamma_A\})$ be an H-space, then for a given nonempty subset K of X, we define the H-convex hull of K, denoted by H-coK as

$$H\text{-}coK = \bigcap\{D \subset X : D \text{ is } H\text{-convex and } D \supset K\}.$$

REMARK. The H-coK is H-convex. Indeed if A is a finite subset of H-coK, then for every H-convex subset D of X with $K \subset D$, $\Gamma_A \subset D$ and hence $\Gamma_A \subset H$-coK. It also follows that H-coK is the smallest H-convex subset containing K.

Now we define the following version of H-space of KKM family in [8];

DEFINITION 1.3. For a nonempty subset D of an H-space X, let $G : D \to 2^X$ be a mapping. A family $\{Gx : x \in D\}$ of subsets of X is called an H-KKM family if $\Gamma_A \subset \bigcup\{Gx : x \in A\}$ for any finite subset A of D. $G : D \to 2^X$ is also called an H-KKM mapping.

In the sequel, we let $\langle D \rangle$ be a family of all finite subsets of D.

DEFINITION 1.4. A subset X_1 of a topological space X is called compactly open (respectively, compactly closed) if every compact set $K \subset X$, the set $X_1 \bigcap K$ is open (respectively, closed) in K.

Now we need the following particular form of the generalized H-KKM theorem due to Chang and Ma[2].

THEOREM 1.1. *Let $(X, \{\Gamma_A\})$ be an H-space and $G : D \to 2^X$ H-KKM mapping with compactly open (closed) values. Then every finite subfamily of $\{Gx : x \in D\}$ has a nonempty intersection.*

Moreover, if G is compactly closed-valued and there exists an $x_0 \in X$ such that $G(x_0)$ is compact, then we have $\bigcap_{x\in X} G(x) \neq \emptyset$.

2. Fixed point theorems on H-spaces

In this section, we prove some fixed point theorems on H-spaces by using the H-KKM theorem.

THEOREM 2.1. *Let $(X, \{\Gamma_A\})$ be an H-space and $S, T : X \to 2^X$ mappings satisfying*

(a) *$Tx \subset Sx$ for each $x \in X$,*

(b) *$S^{-1}y$ is H-convex for each $x \in X$, and*

(c) *$\{X \backslash Tx : x \in X\}$ is not an H-KKM family.*

Then S has a fixed point.

PROOF. Let $Gx \equiv X\backslash Tx$ for each $x \in X$. Since $G : X \to 2^X$ is not an H-KKM mapping, there exists an $A \in \langle X\rangle$ such that $\Gamma_A \not\subset \bigcup\{X\backslash Tx : x \in A\}$. Hence there exists an $x_0 \in \Gamma_A$ such that $x_0 \in Tx \subset Sx$, i.e., $x \in S^{-1}x_0$ for each $x \in A$. Since $S^{-1}x_0$ is H-convex. Thus $\Gamma_A \subset S^{-1}x_0$. Therefore $x_0 \in S^{-1}x_0$, i.e., S has a fixed point.

COROLLARY 2.1. *Let $(X, \{\Gamma_A\})$ be an H-space and $S, T : X \to 2^X$ mappings satisfying*

(a) *$Tx \subset Sx$ for each $x \in X$,*

(b) *$S^{-1}y$ is H-convex for each $x \in X$,*

(c) *Tx is compactly open for each $x \in X$, and there exists an $A \subset \langle X\rangle$ such that $X = \bigcup\{Tx : x \in A\}$.*

Then S has a fixed point.

PROOF. Since $Gx \equiv X\backslash Tx$ is compactly closed, then we have

$$\bigcap\{Gx : x \in A\} = X\backslash \bigcup\{Tx : x \in A\} = \emptyset.$$

By Theorem 1.1, $\{Gx : x \in X\}$ in not an H-KKM family, and hence by Theorem 2.1, S has a fixed point.

COROLLARY 2.2. *Let $(X, \{\Gamma_A\})$ be a compact H-space and $S, T : X \to 2^X$ mappings satisfying*

(a) $Tx \subset Sx$ *for each* $x \in X$,

(b) $S^{-1}y$ *is H-convex for each* $x \in X$,

(c) Tx *is compactly open for each* $x \in X$, *and*

(d) $T^{-1}y \neq \emptyset$ *for each* $x \in X$.

Then S has a fixed point.

PROOF. Since $T^{-1}y \neq \emptyset$ for each $y \in X$ and X is compact. X is covered by a finite number of compactly open sets, say, $Tx_i{}'s (1 \leq i \leq n)$. By Corollary 2.1, S has a fixed point.

REMARK. Theorem 2.1, Corollary 2.1 and Corollary 2.2 are generalizations of Theorems 4, 5 and 6 respectively in [8].

COROLLARY 2.3. *Let $(X, \{\Gamma_A\})$ be an H-space, and $T : X \to 2^X$ a mapping satisfying*

(a) Tx *is compactly open for each* $x \in X$,

(b) $T^{-1}y$ *is H-convex for each* $y \in X$, *and*

(c) X *can be covered by some finite number of sets* Tx_i $(1 \leq i \leq n)$.

Then T has a fixed point.

PROOF. Letting $S = T$ in Corollary 2.1 , we see that T has a fixed point.

REMARK. If we replace the condition (a) in Corollary 2.3 by the following condition $(a)'$, we can have the same result of Corollary 2.3.

$(a)'$ Tx is compactly closed for each $x \in X$.

THEOREM 2.2. *Let $(X, \{\Gamma_A\})$ be an H-space and $T : X \to 2^X$ a mapping such that*

(a) $T^{-1}y$ *is H-convex for each* $y \in X$ *and*

(b) $x \in \Gamma_{\{x\}}$ *for each* $x \in X$.

Then T has a fixed point if and only if $\{X \backslash Tx : x \in X\}$ is not an H-KKM family.

PROOF. (Necessity) Suppose that the conclusion is false, i.e. , $G : X \to 2^X$ be a mapping defined by $G(x) \equiv X \backslash Tx$ for each $x \in X$, then $G : X \to 2^X$ is an H-KKM mapping. Hence for each $x \in X, \{x\} \subset$

$\Gamma_{\{x\}} \subset Gx \equiv X \backslash Tx$. Thus $\{x\} \bigcap Tx = \emptyset$. Hence $x \notin Tx$ for each $x \in X$.

(Sufficiency) Letting $S = T$ in Theorem 2.1. we see that the conclusion holds.

Now we give a generalization of the Fan-Browder fixed point theorems on H-spaces [3] as follows ;

THEOREM 2.3. *Let $(X, \{\Gamma_A\})$ be a compact H-space, and $T : X \to 2^X$ a mapping satisfying*

(a) *$Tx \neq \emptyset$ and H-convex for each $x \in X$,*

(b) *$T^{-1}y$ is compactly open for each $y \in X$.*

Then T has a fixed point.

PROOF. Let $G(x) \equiv X \backslash T^{-1}y$. Then $G : X \to 2^X$ is not an H-KKM mapping. In fact, if G is an H-KKM mapping, then by Theorem 1.1, $\bigcap\{Gy : y \in X\} \neq \emptyset$, i.e., there exists a $y_0 \in \bigcap\{Gy : y \in X\}$ then $y_0 \in Gy$ for each $y \in X$. Thus $y_0 \in X \backslash T^{-1}y$ for each $y \in X$. Then $y_0 \notin T^{-1}y$, i.e., $y \notin Ty_0$ for each $y \in X$, then $Ty_0 = \emptyset$. This is a contradiction. By Theorem 2.2 , T has a fixed point.

DEFINITION 2.1. Let X and Y be two topological spaces, then a mapping $T : X \to 2^Y$ is said to be upper semi-continuous (for short, u.s.c.) if for each open subset V of Y, the set $\{x \in X : T(x) \subset V\}$ is open in X.

COROLLARY 2.4. *Let $(X, \{\Gamma_A\})$ be an H-space, and $T : X \to 2^X$ a mapping satisfying*

(a) *$T : X \to 2^X$ is u.s.c.,*

(b) *Tx is H-convex for each $x \in X$, and*

(c) *there exists a finite subset K of X such that $Tx \bigcap K \neq \emptyset$ for each $x \in X$.*

Then T has a fixed point.

PROOF. Let $G : X \to 2^X$ be a mapping defined by $Gx \equiv T^{-1}x = \{y \in X : x \in Ty\}$ for each $x \in X$. Since T is u.s.c., Gx is closed and hence Gx is compactly closed for each $x \in X$. And we have $G^{-1}y = (T^{-1})^{-1}y = Ty$ is H-convex. Hence X can be coverd by a finite subfamily of $\{Gx : x \in X\}$. By the Remark following Corollary 2.3, there exists $y_0 \in X$ such that $y_0 \in Gy_0$, i.e., $y_0 \in Ty_0$.

REMARK. Corollary 2.4 is a generalization of Corollary 1 in [5].

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