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ON CERTAIN SUBCLASSES OF STARLIKE FUNCTIONS

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ABSTRACT. The class $R_{\gamma-1,p}(A,B,\alpha)$ for $-1 \le B < A \le 1, \gamma > (B-1)p+(A-B)(p-\alpha)/1-B$ and $0 \le \alpha < p$ consisting of p-valently analytic functions in the open unit disc is defined with the help of convolution technique. We study containment property, integral transforms and a sufficient condition for an analytic function to be in $R_{\gamma-1,p}(A,B,\alpha)$.

1. Introduction

Let $A(p)$ denote the class of functions $f(z) = z^p + a_{p+1}z^{p+1} + a_{p+2}z^{p+2} + \cdots$, a positive integer p which are regular in the unit disc $E = \{z : |z| < 1\}$. If $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k$ and $g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k$ belong to $A(p)$, we define the Hadamard product or convolution of f and g by $(f*g)(z) = z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k, z \in E$.

Let A, B, γ and α be arbitrary fixed real numbers such that $-1 \le B < A \le 1, \gamma > (B-1)p + (A-B)(p-\alpha)/(1-B)$ and $0 \le \alpha < p$, we say that a function f in $A(p)$ is in the class $R_{\gamma-1,p}(A,B,\alpha)$ if it satisfies the condition

$$(1.1) \qquad \frac{z(D^{\gamma+p-1}f(z)')}{D^{\gamma+p-1}f(z)} \prec \frac{p + \{pB + (A-B)(p-\alpha)\}z}{1+Bz}, \quad z \in E.$$

where$'$ denotes the differentiation with respect to z, $\prec$ denotes the subordination and $D^{\gamma+p-1}f(z) = z^p/(1-z)^{\gamma+p} * f(z)$. Let B_0 denote the

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class of analytic functions $w(z)$ in E such that $w(0) = 0$ and $|w(z)| < |z|$. The equation (1.1) becomes

$$\frac{z(D^{\gamma+p-1}f(z))'}{D^{\gamma+p-1}f(z)} = \frac{p + \{pB + (A-B)(p-\alpha)\}w(z)}{1 + Bw(z)} \tag{1.2}$$

for some $w(z) \in B_0$.

It can be seen that

$$z(D^{\gamma+p-1}f(z))' = (\gamma + p)D^{\gamma+p}f(z) - \gamma D^{\gamma+p-1}f(z). \tag{1.3}$$

From this, the equation (1.2) is equivalent to

$$\frac{D^{\gamma+p}f(z)}{D^{\gamma+p-1}f(z)} = \frac{\gamma}{\gamma+p} + \frac{1}{\gamma+p}\frac{p + \{pB + (A-B)(p-\alpha)\}w(z)}{1 + Bw(z)}.$$

LEMMA 1. *If a function $f(z)$ is in $R_{\gamma-1,p}(A, B, \alpha)$, then it satisfies*

$$\left|\frac{z(D^{\gamma+p-1}f(z))'}{D^{\gamma+p-1}f(z)} - \frac{p - \{pB + (A-B)(p-\alpha)\}B}{1 - B^2}\right| < \frac{(A-B)(p-\alpha)}{1-B^2}$$

for $z \in E$, $B \neq -1$,

and

$$Re\left\{\frac{z(D^{\gamma+p-1}f(z))'}{D^{\gamma+p-1}f(z)}\right\} > \frac{2p - (A+1)(p-\alpha)}{2} \quad \text{for } z \in E,\ B = -1.$$

We can consider $R_\gamma(A, B)$ and $R_n(1 - 2\beta, -1)$ as the special case of $R_{\gamma-1,p}(A, B, \alpha)$. For $p = 1$ and $\alpha = 0, R_\gamma(A, B)$ has been studied by T. Ram Reddy[3] and for any $\beta(0 \leq \beta < 1), \gamma = n \in N_0$, we have $R_n(1 - 2\beta, -1)$ which was studied by Ahuja and Silverman[1]. In this paper, we shall establish the containment property $R_{\gamma,p}(A, B, \alpha) \subset R_{\gamma-1,p}(A, B, \alpha)$, and also study some other aspects such as integral transforms, sufficient condition of functions belonging to the class $R_{\gamma-1,p}$ (A, B).

2. Main Result

THEOREM 2.1. *$R_{\gamma,p}(A,B,\alpha) \subset R_{\gamma-1,p}(A,B,\alpha)$ hold for $-1 \leq B < A \leq 1$, $\gamma > \frac{(B-1)p+(A-B)(p-\alpha)}{1-B}$ and $0 \leq \alpha < p$.*

PROOF. Suppose $f(z)$ is in $R_{\gamma,p}(A,B,\alpha)$. Then we have

$$\begin{aligned}(2.1)\qquad &\left|\frac{z(D^{\gamma+p}f(z))'}{D^{\gamma+p}f(z)} - \frac{p-\{pB+(A-B)(p-\alpha)\}B}{1-B^2}\right| \\ &< \frac{(A-B)(p-\alpha)}{1-B^2} \qquad \text{for } z \in E, B \neq -1,\end{aligned}$$

and

$$\begin{aligned}(2.2)\qquad &\mathrm{Re}\left\{\frac{z(D^{\gamma+p}f(z))'}{D^{\gamma+p}f(z)}\right\} > \frac{2p-(A+1)(p-\alpha)}{2} \\ &\text{for } z \in E, B = -1.\end{aligned}$$

Define a function $w(z)$ such that

$$(2.3)\qquad \frac{z(D^{\gamma+p}f(z))'}{D^{\gamma+p}f(z)} = \frac{p+\{pB+(A-B)(p-\alpha)\}w(z)}{1+Bw(z)}.$$

The function

$$w(z) = \frac{\frac{z(D^{\gamma+p-1}f(z))'}{D^{\gamma+p-1}f(z)} - p}{\{pB+(A-B)(p-\alpha)\} - B\cdot\left\{\frac{z(D^{\gamma+p-1}f(z))'}{D^{\gamma+p-1}f(z)}\right\}},$$

with $w(0) = 0, w(z) \neq 1$, is either meromorphic or regular in E. It is sufficient to show that $|w(z)| < 1$ for $z \in E$. Using the identity (1.3), the equation (2.3) reduces to

$$(2.4)\qquad \frac{D^{\gamma+p}f(z)}{D^{\gamma+p-1}f(z)} = \frac{\gamma}{\gamma+p} + \frac{1}{\gamma+p}\cdot\frac{p+\{pB+(A-B)(p-\alpha)\}w(z)}{1+Bw(z)}.$$

Differentiating (2.4) and using (2.3) and (1.3) we obtain

(2.5)
$$\begin{aligned}&\frac{z(D^{\gamma+p}f(z))'}{D^{\gamma+p}f(z)}\\ =&\frac{p+\{pB+(A-B)(p-\alpha)\}w(z)}{1+Bw(z)}\\ &+\frac{(A-B)(p-\alpha)zw'(z)}{[(\gamma+p)+\{pB+(A-B)(p-\alpha)+B\gamma\}w(z)]\cdot(1+Bw(z))}.\end{aligned}$$

Case (i). If $B\neq-1$, then we have from (2.5),

(2.6)
$$\begin{aligned}&\frac{z(D^{\gamma+p}f(z))'}{D^{\gamma+p}f(z)}-\frac{p-\{pB+(A-B)(p-\alpha)\}B}{1-B^2}\\ =&\frac{(A-B)(p-\alpha)}{1-B^2}\cdot\frac{B+w(z)}{1+Bw(z)}\\ &+\frac{(A-B)(p-\alpha)zw'(z)}{[(\gamma+p)+\{pB+(A-B)(p-\alpha)+B\gamma\}w(z)]\cdot(1+Bw(z))}.\end{aligned}$$

Let z_1 with $|z_1|=r_1$ be the pole of $w(z)$ in E that is nearest to the origin. Hence $w(z)$ is regular in $|z|\leq r_1<1$. By Jack's lemma[2], we have for $|z|\leq r<r_1$ there is a point z_0 such that

(2.7)
$$z_0w'(z_0)=kw(z_0),\qquad k\geq1.$$

At this point z_0, using (2.7) the equation (2.6) reduces to

(2.8)
$$\begin{aligned}&\frac{z(D^{\gamma+p}f(z_0))'}{D^{\gamma+p}f(z_0)}-\frac{p-\{pB+(A-B)(p-\alpha)\}B}{1-B^2}\\ =&\frac{T(z_0)}{R(z_0)}\frac{(A-B)(p-\alpha)}{1-B^2},\end{aligned}$$

where

(2.9)
$$T(z_0)=d+hw(z_0)+gw^2(z_0),\ \text{and}$$

(2.10)
$$R(z_0)=(\gamma+p)+(d+g)w(z_0)+\frac{dg}{\gamma+p}w^2(z_0)\quad\text{with}$$

$d = B(\gamma + p), h = (\gamma + p) + B\{pB + (A - B)(p - \alpha) + B\gamma\} + k(1 - B^2)$ and $g = B(p + \gamma) - (A - B)(p - \alpha)$.

Now suppose that it was possible to have $\mathcal{M}(r, \omega) = \max |\omega(z)| = 1$, for some $r < r_1 < 1$. At point z_0 where this occured we would have $|\omega(z)| = 1$, then

$$
\begin{aligned}
|T(z_0)|^2 =& d^2 + h^2 + g^2 + 2(d + g)h \cdot \mathrm{Re} \quad w(z_0) \\
& + 2dg \cdot \mathrm{Re} \quad w^2(z_0)
\end{aligned}
\tag{2.11}
$$

and

$$
\begin{aligned}
|R(z_0)|^2 =& (\gamma + p)^2 + (d + g)^2 + \left(\frac{dg}{\gamma + p}\right)^2 \\
& + 2\left\{(\gamma + p) + \frac{dg}{(\gamma + p)}\right\} \cdot (d + g) \cdot \mathrm{Re} \quad w(z_0) \\
& + 2dg \cdot \mathrm{Re} \quad w^2(z_0).
\end{aligned}
\tag{2.12}
$$

From (2.11) and (2.12) we have

$$
|T(z_0)|^2 - |R(z_0)|^2 = E + 2F \cdot \mathrm{Re} \quad w(z_0),
\tag{2.13}
$$

where

$$
\begin{aligned}
E =& d^2 + h^2 + g^2 - [(\gamma + p)^2 + (d + g)^2 + (\frac{dg}{\gamma + p})^2] \\
=& k(1 - B^2)[k(1 - B^2) + 2\{(\gamma + p) \\
& + B\{B(\gamma + p) + (A - B)(p - \alpha)\}\}], \\
F =& (d + g)h - [(\gamma + p) \cdot (d + g) + \frac{dg}{\gamma + p}(d + g)] \\
=& [B(\gamma + p) + pB + (A - B)(p - \alpha) + B\gamma]k(1 - B^2).
\end{aligned}
$$

Thus

$$
|T(z_0)|^2 - |R(z_0)|^2 \geq 0 \quad \text{provied } E \pm 2F \geq 0.
\tag{2.14}
$$

Now we have

$$\begin{aligned}E+2F=&k((1-B^2)[k(1-B^2)\\&+2\{(\gamma+p)+pB+(A-B)(p-\alpha)+B\gamma\}(1+B)]>0,\end{aligned}$$

since $\gamma > \frac{(B-1)p+(A-B)(p-\alpha)}{1-B}$.

Also we obtain

$$\begin{aligned}E-2F=&k(1-B^2)[k(1-B^2)\\&+2\{(\gamma+p)-\{pB+(A-B)(p-\alpha)+B\gamma\}(1-B)\}]>0,\end{aligned}$$

since $\gamma > \frac{(B-1)p+(A-B)(p-\alpha)}{1-B}$.

From (2.8) and (2.14) we have

$$\left|\frac{z_0(D^{\gamma+p}f(z))'}{D^{\gamma+p}f(z)}-\frac{p-\{pB+(A-B)(p-\alpha)\}B}{1-B^2}\right| \geq \frac{(A-B)(p-\alpha)}{1-B^2},$$

which is a contradiction to (2.1). Thus $|w(z)| \neq 1$ in $|z| < r_1$. Since by our assumption $w(z)$ is analytic in $|z| < r_1$, $|w(z)|$ is continuous there. Again $w(0) = 0$ and $|w(z)| \neq 1$. Hence $w(z)$ cannot have a pole at $|z| = r_1$. As r_1 is arbitrary, $w(z)$ is analytic in E and $|w(z)| < 1$. Hence $f(z)$ is in $R_{\gamma-1,p}(A, B, \alpha)$.

Case (ii). If $B = -1$, then (2.5) reduces to

$$\begin{aligned}&\frac{z(D^{\gamma+p}f(z))'}{D^{\gamma+p}f(z)}\\ (2.15)\qquad &=\frac{p+\{-p+(A+1)(p-\alpha)\}w(z)}{1-w(z)}\\&+\frac{(A+1)(p-\alpha)zw'(z)}{[(\gamma+p)+\{-p+(A+1)(p-\alpha)-\gamma\}w(z)](1-w(z))}.\end{aligned}$$

We claim again that $|w(z)| < 1$. Suppose if possible that there is a point z_0 in E such that $\max_{|z|\leq|z_0|}|\omega(z)| = |\omega(z_0)| = 1$, then by Jack's lemma[2], we have $z_0 w'(z_0) = kw(z_0), k \geq 1$. Hence at the point z_0 with $w(z_0) = e^{i\phi}$ $(0 \leq \phi < 2\pi)$ we have from (2.15).

$$\begin{aligned}
&\operatorname{Re}\left\{\frac{z_0(D^{\gamma+p}f(z_0))'}{D^{\gamma+p}f(z_0)}\right\} \\
&= \operatorname{Re}\left[\frac{p+\{-p+(A+1)(p-\alpha)\}e^{i\phi}}{1-e^{i\phi}}\right. \\
&\quad \left. + \frac{(1+A)(p-\alpha)ke^{i\phi}}{\{(\gamma+p)+\{-p+(A+1)(p-\alpha)-\gamma\}e^{i\phi}\}(1-e^{i\phi})}\right] \\
&= \frac{-(A+1)(p-\alpha)+2p}{2} + k\left[\frac{-1}{2} + \right. \\
&\quad \left. \frac{\{(A+1)(p-\alpha)+(p+\gamma)\}^2+(\gamma+p)\{(A+1)(p-\alpha)-(p+\alpha)\}\cos\theta}{(\gamma+p)^2+\{(A+1)(p-\alpha)-(p+\gamma)\}^2+2(\gamma+p)\{(A+1)(p-\alpha)-(p+\gamma)\}\cos\theta}\right] \\
&< \frac{-(A+1)(p-\alpha)+2p}{2}, \quad \text{since } k \geq 1 \text{ and } \gamma > \frac{-2p+(A+B)(p-\alpha)}{2},
\end{aligned}$$

which is again a contradiction to (2.2). Thus we must have $|w(z)| < 1$. Hence the theorem is true for $B = -1$ also. □

THEOREM 2.2. *If $g(z)$ is in $R_{\gamma-1,p}(A,B,\alpha), \gamma+p-1 \geq 0$ and $G(z)$ is defined by*

$$G(z) = (c+p)\int_0^1 u^{c-1} g(uz)\,du, \quad \text{for} \quad c > \frac{(B-1)p+(A-B)(p-\alpha)}{1-B}, \tag{2.16}$$

then $G(z)$ is in $R_{\gamma-1,p}(A,B,\alpha)$.

PROOF. Suppose $g(z)$ is in $R_{\gamma-1,p}(A,B,\alpha)$, then we have

$$\left|\frac{z(D^{\gamma+p-1}g(z))'}{D^{\gamma+p-1}g(z)} - \frac{p-\{pB+(A-B)(p-\alpha)\}B}{1-B^2}\right| < \frac{(A-B)(p-\alpha)}{1-B^2} \quad \text{if } B \neq -1 \text{ for } z \in E, \tag{2.17}$$

and

$$\text{(2.18)} \qquad \operatorname{Re}\left\{\frac{z(D^{\gamma+p-1}g(z))}{D^{\gamma+p-1}g(z)}\right\} > \frac{2p-(A+1)(p-\alpha)}{2}$$
$$\text{if } B=-1 \text{ for } z\in E.$$

From the definition of $G(z)$, we have

$$\text{(2.19)} \qquad zG'(z)+cG(z)=(c+p)g(z).$$

Taking convolution on both side $\frac{z^p}{(1-z)^{\gamma+p}}$ for $\gamma+p-1\geq 0$, we get

$$\text{(2.20)} \qquad z(D^{\gamma+p-1}G(z))'+c(D^{\gamma+p-1}G(z))=(c+p)D^{\gamma+p-1}g(z).$$

Thus

$$\text{(2.21)} \qquad \frac{z(D^{\gamma+p-1}G(z))'}{D^{\gamma+p-1}G(z)}+c=(c+p)\frac{D^{\gamma+p-1}g(z)}{D^{\gamma+p-1}G(z)}.$$

Since $h(z)=\frac{z(D^{\gamma+p-1}G(z))'}{D^{\gamma+p-1}G(z)}$ is analytic at $z=0$, we can define a function $w(z)$ such that

$$\text{(2.22)} \qquad \frac{z(D^{\gamma+p-1}G(z))'}{D^{\gamma+p-1}G(z)}=\frac{p+\{pB+(A-B)(p-\alpha)\}w(z)}{1+Bw(z)}.$$

The function

$$w(z)=\frac{\frac{z(D^{\gamma+p-1}G(z))'}{D^{\gamma+p}G(z)}-p}{\{pB+(A-B)(p-\alpha)\}-B\cdot\frac{z(D^{\gamma+p-1}G(z))'}{D^{\gamma+p-1}G(z)}},$$

with $w(0)=0, w(z)\neq 1$ is either analytic or meromorphic in E. It is sufficient to show that $|w(z)|<1$ for $z\in E$.

From (2.21) and (2.22) we have

$$\text{(2.23)} \qquad \begin{aligned}&\frac{D^{\gamma+p-1}g(z)}{D^{\gamma+p-1}G(z)}\\ =&\frac{1}{c+p}\left\{\frac{(c+p)+\{pB+(A-B)(p-\alpha)+Bc\}w(z)}{1+Bw(z)}\right\}.\end{aligned}$$

Differentiating (2.23) logarithmically, using (2.22) and (1.3) we have

$$
\begin{aligned}
&\frac{z(D^{\gamma+p-1}g(z))'}{D^{\gamma+p-1}g(z)} \\
(2.24) \quad =&\frac{p+\{pB+(A-B)(p-\alpha)\}w(z)}{1+Bw(z)} \\
&+\frac{(A-B)(p-\alpha)zw'(z)}{[(c+p)+\{pB+(A-B)(p-\alpha)+Bc\}w(z)](1+Bw(z))}.
\end{aligned}
$$

We observe that the right hand side of (2.24) is essentially same as that of (2.5) except that γ is replaced by c in the second term. Thus proceeding exactly in similar manner as in the proof of Therem 2.1, we have $G(z)\in R_{\gamma-1,p}(A,B,\alpha)$.

When $\alpha=0$ and $p=1$ we obtain a result due to T. Ram Reddy and O.P Juneja [3].

COROLLARY 2.3. *If $g(z)$ is in $R_\gamma(A,B)$, then the function $G(z)$ defined by* (2.16) *is also in $R_\gamma(A,B)$.*

Again putting $\alpha=0, p=1, A=1, B=-1$ and $\gamma\in N_0$ we obtain a result due to [4].

COROLLARY 2.4. *If $g(z)$ is in $R_n(1,-1)$, then the function $G(z)$ defined by* (2.16) *is also in $R_n(1,-1)$.*

THEOREM 2.5. *For a function $f(z)=z^p+\sum_{k=p+1}a_kz^k$ in $A(p)$ if for some real number $\gamma\geq 0, 0\leq\alpha<p$ and A,B with $-1\leq B<A\leq 1$,*

$$
\begin{aligned}
(2.25) \quad &\sum_{k=p+1}^{\infty}\{(k-p)+|pB+(A-B)(p-\alpha)-B_k|\}C(\gamma,p,k)|a_k| \\
&\leq(A-B)(p-\alpha)-(p-1),
\end{aligned}
$$

then $f(z)$ belongs to the class $R_{\gamma-1,p}(A,B,\alpha)$, where

$$
(2.26) \qquad C(\gamma,p,k)=\frac{\Pi_{j=1}^{k-p}(\gamma+p-1+j)}{(k-p)!}.
$$

PROOF. Suppose that (2.25) hold. Since $D^{\gamma+p-1}f(z) = z^p + \sum_{k=p+1}^{\infty} C(\gamma,p,k)a_k z^k$, we have $z \in E$,

$$\begin{aligned}
&|z(D^{\gamma+p-1}f(z))' - pD^{\gamma+p-1}f(z)| \\
&\quad - |\{pB + (A-B)(p-\alpha)\}D^{\gamma+p-1}f(z) - Bz(D^{\gamma+p-1}f(z)'| \\
\leq &(p-1)r^p + \sum_{k=p+1}^{\infty} (k-p)C(\gamma,k,p)|a_k|r^k \\
&\quad - \Big\{(A-B)(p-\alpha)r^p - \sum_{k=p+1}^{\infty} |pB + (A-B)(p-\alpha) \\
&\quad - B_k|C(\gamma,k,p)|a_k|r^k\Big\} \\
= &[(p-1) - (A-B)(p-\alpha) \\
&\quad + \sum_{k=p+1}^{\infty} \{(n-p) + |pB + (A-B)(p-\alpha) - B_k|\}C(\gamma,k,p)|a_k|]r \\
< &0 \quad \text{by } (2.25).
\end{aligned}$$

Hence it follows that

$$\left| \frac{\frac{z(D^{\gamma+p-1}f(z))'}{D^{\gamma+p-1}G(z)} - p}{pB + (A-B)(p-\alpha) - B \cdot \frac{z(D^{\gamma+p-1}f(z))'}{D^{\gamma+p-1}f(z)}} \right| < 1. \tag{2.27}$$

Let $w(z)$ denote the term inside the modulus of (2.27), then $w(0) = 0$, $w(z)$ is analytic in $|z| < 1$ and $|w(z)| < 1$. Hence we have

$$\frac{z(D^{\gamma+p-1}f(z))'}{D^{\gamma+p-1}f(z)} = \frac{p + \{pB + (A-B)(p-\alpha)\}w(z)}{1 + Bw(z)}$$

which shows that $f(z)$ belongs to the class $R_{\gamma-1,p}(A,B,\alpha)$.

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