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# CONVERGENCE OF NONEXPANSIVE MAPPINGS

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ABSTRACT. Our main purpose in this paper is to discuss a new result on convergence of nonlinear contractive algorithms acting on different Banach spaces.

## 1. Intorduction

In this paper we intend to prove a general result on the approximation and convergence of nonexpansive mappings acting on different Banach space. Such convergence problems arise naturally in many settings, including, for example, the numerical treatment of partial differential equations and parameter identification theory. In particular, we will present new version of nonlinear Chernoff type theorem. This new result are useful in obtaining approximations of solutions to differential equations via difference scheme (see [4]). Our Chernoff type theorem is a genralization presented by Y. S. Lee and S. Reich [4]. It also includes the one space result [5].

Let $X$ and $X_n$ be Banach spaces with norm $|\cdot|$ and $|\cdot|_n$, respectively. For every $n \geq 1$ there exist bounded linear operators $P_n : X \to X_n$ and $E_n : X_n \to X$ satisfying

(1) $||P_n|| \leq 1$ and $||E_n|| \leq 1$ for all $n$,
(2) $|E_n P_n x - x| \to 0$ as $n \to \infty$ for all $x \in X$,
(3) $P_n E_n = I_n$, where $I_n$ is the identity operator on $X_n$.

Note that we do not assume that the spaces $X_n$ are subspaces of $X$. See [1, 7] for related setting. The introduction of $X_n$, $P_n$ and $E_n$ is motivated by the approximation of differential equations via difference equations, since the differential equations act on spaces different from

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the one on which the difference operator acts. For examples of $P_n$ and $E_n$, see [4].

## 2. Main Theorems

Let $B$ be a subset of a Banach space $(X, |\cdot|)$. Recall that a mapping $T : B \to X$ is said to be nonexpansive if $|Tx - Ty| \leq |x - y|$ for all $x,\ y \in B$.

THEOREM. *Let $D$ be a closed convex subset of $X$. Let $\{\rho_n\}$ be a sequence of positive numbers converging to 0. Let $F(\rho_n)$ be nonexpansive mappings from a closed convex subset $C_n$ of $X_n$ into itself such that $P_n(D) \subset C_n$ and $E_n(C_n) \subset D$ for all $n$.*

*Suppose that*

$$\text{(I)} \qquad \lim_{n \to \infty} E_n(I + \frac{r}{\rho_n}(I - F(\rho_n)))^{-1} P_n x = J_r x$$

*exists for each $x \in D$ and $r > 0$, and $C = cl\{J_r x : x \in D \text{ and } r > 0\}$. If $\{k_n\}$ is a sequence of positive integers such that $\lim_{n \to \infty} k_n \rho_n = t$, then*

$$\text{(II)} \qquad \lim_{n \to \infty} E_n F(\rho_n)^{k_n} P_n x = S(t)x$$

*exist for all $x \in C$ and $t \geq 0$, and the convergence is uniform on bounded $t$-intervals.*

This theorem shows that the resolvent consistency (I) is a sufficient condition for convergence of the nonlinear product formula (II). The limit $J_r$ in (I) is the resolvent of an accretive operator $A$ satisfying the range condition and the limit $S$ in (II) is the nonlinear semigroup generated by $-A$ on $C$. For definitions and notions related to accretive operators and semigroups, we refer the reader to [2]. We note that we let $F(0)$ be the identy in X. This is reasonable in view of the following observation.

LEMMA. *If (I) holds, then $\lim_{n \to \infty} E_n F(\rho_n) P_n y = y$ for each $y \in C$.*

PROOF. Assume that $y = J_r x$ for some $x \in D$. Let $y_n = E_n(I + \frac{r}{\rho_n}(I - F(\rho_n)))^{-1} P_n x$. Then $\lim_{n\to\infty} y_n = y$. So we have $\lim_{n\to\infty}(y_n - E_n F(\rho_n) P_n y_n) = 0$. Therefore we have

$$\begin{aligned}
&|E_n F(\rho_n) P_n y - y| \\
&\quad \le |E_n F(\rho_n) P_n y - E_n F(\rho_n) P_n y_n| + |E_n F(\rho_n) P_n y_n - y_n| + |y_n - y| \\
&\quad \le 2|y_n - y| + |E_n F(\rho_n) P_n y_n - y_n| \to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

Now let $y \in C$. Then there exist $y_k = J_{r_k} x_k$ such that $\lim_{k\to\infty} y_k = y$. Then

$$\begin{aligned}
&|E_n F(\rho_n) P_n y - y| \\
&\quad \le |E_n F(\rho_n) P_n y - E_n F(\rho_n) P_n y_k| + |E_n F(\rho_n) P_n y_k - y_k| + |y_k - y| \\
&\quad \le 2|y_k - y| + |E_n F(\rho_n) P_n y_k - y_k|
\end{aligned}$$

Let $\varepsilon > 0$ be given. Choose $k$ such that $|y - y_k| < \varepsilon/3$ and then choose $n_0$ such that $|E_n F(\rho_n) P_n y_k - y_k| < \varepsilon/3$.

PROOF OF THEOREM. We define an operator $A \subset X \times X$ by

$$A = \bigcup_{r>0} \{[J_r, \ \frac{1}{r}(x - J_r x)] : x \in D\}.$$

To prove theorem, it is enough to show that $A$ is an accretive operator satisfying the range condition $R(I + rA) \supset cl(D(A))$ for all $r > 0$ and the limit $J_r$ in (I) is the resolvent of $A$. (For more information and complete proof, see [4].)

Let $J_{r,n} = (I + \frac{r}{\rho_n}(I - F(\rho_n)))^{-1} P_n$. Then $J_{r,n}$ is nonexpansive. In order to show that $A$ is accretive, it is sufficient to show that $J_r$ satisfies the resolvent identity (see [3]), that is,

$$J_r x = J_s(\frac{s}{r} x + \frac{r-s}{r} J_r x) \quad \text{for} \quad r, \ s > 0.$$

It is not difficult to show that

$$J_{r,n} P_n x = J_{s,n}(\frac{s}{r} P_n x + \frac{r-s}{r} J_{r,n} P_n x).$$

Consider

$$\begin{aligned}&|E_nJ_{s,n}(\frac{s}{r}P_nx+\frac{r-s}{r}J_{r,n}P_nx)-J_s(\frac{s}{r}x+\frac{r-s}{r}J_rx)|\\&\quad\le|E_nJ_{s,n}(\frac{s}{r}P_nx+\frac{s-r}{r}J_{r,n}P_nx)-E_nJ_{s,n}P_n(\frac{s}{r}x+\frac{s-r}{r}J_rx)|\\&\quad+|E_nJ_{s,n}P_n(\frac{s}{r}x+\frac{s-r}{r}J_rx)-J_s(\frac{s}{r}x+\frac{s-r}{r}J_rx)|\to 0\end{aligned}$$

as $n\to\infty$. Hence we can conclude that $J_r$ satisfies the resolvent identity. Thus the operator $A$ is accretive, $R(I+rA)\supset D$ for each $r>0$, and $cl(D(A))=C$. Therefore $-A$ generates a nonlinear semigroup $S$ on $C$. The resolvent $(I+rA)^{-1}$ of $A$ on $D$ coincides with the limit $J_r$ in (I) (see [3]).

REMARK. In the one space case, that is, $X_n=X$, $P_n=E_n=I$ for all $n$, our theorem includes theorem 1 in [5].

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