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INVARIANTS OF THE SYMMETRIC GROUP

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ABSTRACT. Let $R = k[y_1, \cdots, y_n] \otimes E[x_1, \cdots, x_n]$ with characteristic $k = p > 2$ (odd prime), where $|y_i| = 2, |x_i| = 1$ and $y_i = \beta x_i$, β is the Bockstein homomorphism. Topologically, $R = H^*(B(Z/p)^n, k)$. For a symmetric group Σ_n, $R^{\Sigma_n} = k[\sigma_1, \cdots, \sigma_n] \otimes E[d\sigma_1, \cdots, d\sigma_n]$ where d is the derivation satisfying $d(y_i) = x_i$ and $d(x_i y_j) = x_i y_j + x_j y_i$, $1 \leq i,, j \leq n$. We give a direct proof of this theorem by using induction.

1. Introduction

Let V be an n dimensional vector space with basis $\{x_1, \cdots, x_n\}$ over a field k of characteristic p (prime) and G a finite subgroup of $GL(V)$. Then G acts on the symmetric algebra $\bar{R} = k[x_1, \cdots, x_n]$ by algebra automorphism i.e. $(gf)(v) = f(g^{-1}v)$ for $g \in G, f \in \bar{R}$ and $v \in V$. We denote by $\bar{R}^G$ the subalgebra of $\bar{R}$ consisting of all invariant polynomials under the action of G. For the symmetric algebra $\bar{R}$, where characteristic $k = p \geq 0$, invariants are familiar with two cases. One is $p = 0$ or $p \nmid |G|$, so-called nonmodular case. The other is $p \mid |G|$, the modular case. For the nonmodular case, Emmy Noether [7] showed that when $p = 0$, $\bar{R}^G$ is generated by $\binom{|G|+n}{n}$ polynomials of degree at most $|G|$. Later, in 1991, H. E. A. Campbell, I. Hughes and R. D. Pollack [1] generalized this theorem to arbitrary characteristic at least nonmodular groups. Their result is that for $p \nmid |G|$, $\bar{R}^G$ is generated by the polynomials with degree at most $\max(|G|, n\binom{|G|}{2})$. With this result, the author generalized this theory to the tensor product of a polynomial algebra and an exterior algebra. We denote $R = k[y_1, \cdots, y_n] \otimes E[x_1, \cdots, x_n]$ where characteristic $k = p > 2, |y_i| = 2, |x_i| = 1$ and $y_i = \beta x_i, \beta$ is the Bockstein homomorphism. Topologically $R = H^*(B(\mathbf{Z}/p)^n, k)$. The author proved the

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following theorem in [3] ; If G is a nontrivial group and p is an odd prime such that $p \nmid |G|$, then R^G is generated by polynomials with degree at most $n(|G|^2 - 1)$ if $n > 1$ or at most $2|G|$ if $n = 1$. For the modular case, the ring of invariants of the full linear group $GL(V)$ of a finite dimensional vector space over the finite field $\mathbf{F}_p$ was computed early in the 20th century by L. E. Dickson [2], and was found to be a graded polynomial algebra on certain generators $\{c_{n,i}\}$. In 1975, H. Mui [4] generalized Dickson's theory to the algebra of $k[y_1, \cdots, y_n : 2] \otimes E[x_1, \cdots, x_n : 1]$ under $GL(V)$ and $GL_{n,p}$ (a Sylow p-subgroup consisting of upper triangular matrices). For a symmetric group Σ_n, the theorem has been known that $R^{\Sigma_n} = k[\sigma_1, \cdots, \sigma_n] \otimes E[d\sigma_1, \cdots, d\sigma_n]$ as the image of the map of de Rham complexes by D. Quillen [5]. In this paper, we give a direct proof of this theorem by using induction. More generally, L. Solomon [6] gave the structure of the ring of invariants for the finite reflection groups over R.

2. Proof of the Theorem

We set $R = k[y_1, \cdots, y_n : 2] \otimes E[x_1, \cdots, x_n : 1]$ with characteristic $k = p > 2$, where $|y_i| = 2, |x_i| = 1$ and $y_i = \beta x_i$, β is the Bockstein homomorphism. Let $\alpha = y_1^{a_1} \cdots y_n^{a_n} x_1^{\epsilon_1} \cdots x_n^{\epsilon_n}$ be a monomial of R where $a_i \in N$ (nonnegative integers) and $\epsilon_i = 0$ or 1. Σ_n acts on α by $\sigma(\alpha) = y_{\sigma(1)}^{a_1} \cdots y_{\sigma(n)}^{a_n} x_{\sigma(1)}^{\epsilon_1} \cdots x_{\sigma(n)}^{\epsilon_n}$ for $\sigma \in \Sigma_n$. We denote $\sigma(\alpha)$ by α_σ. We set $(\Sigma_n)_\alpha = \Sigma_n / Stab(\alpha)$ where $Stab(\alpha) = \{\sigma \in \Sigma_n \mid \sigma(\alpha) = \alpha\}$. We set $s(\alpha) = \sum_{\sigma \in (\Sigma_n)_\alpha} \alpha_\sigma$. Then we observe $s(\alpha) \in R^{\Sigma_n}$ for all α. Denote the degree of α by $|\alpha|$.

DEFINITION 2.1. A polynomial in R which is unchanged by any permutation of the indeterminates $y_1, \cdots, y_n : x_1, \cdots, x_n$ is called a symmetric function.

Let $\{\sigma_i\}_1^n$ be the elementary symmetric functions of the variables $y_1, \cdots, y_n$. Let d be the derivation satisfying $d(y_i) = x_i$ and $d(y_i y_j) = x_i y_j + y_i x_j$, $1 \le i, j \le n$. For example, $d\sigma_1 = x_1 + x_2 + \cdots + x_n$, $d\sigma_2 = x_1 y_2 + x_2 y_1 + \cdots + x_{n-1} y_n + x_n y_{n-1}, \cdots\cdots, d\sigma_n = x_1 y_2 y_3 \cdots y_n + \cdots + x_n y_1 \cdots y_{n-1}$. The polynomials σ_i and $d\sigma_i$ are homogeneous of degree $2i$ and $2i - 1$ respectively. A polynomial $\psi(\sigma_1, \cdots, \sigma_n : d\sigma_1, \cdots, d\sigma_n)$ becomes a symmetric function of $y_1, \cdots, y_n : x_1, \cdots, x_n$ where $d\sigma$ are written by x and y, and σ are written in terms of y. Therefore a term $c\sigma_1^{\mu_1}\cdots\sigma_n^{\mu_n}d\sigma_1^{\epsilon_1}\cdots d\sigma_n^{\epsilon_n}$ of a polynomial $\psi(\sigma_1,\cdots,\sigma_n : d\sigma_1,\cdots,d\sigma_n)$ becomes a homogeneous polynomial in the x_i and y_i of degree $2\sum_{i=1}^n i\mu_i + \sum_{i=1}^n(2i-1)\epsilon_i$. We call this sum *weight* of the terms $c\sigma_1^{\mu_1}\cdots\sigma_n^{\mu_n}d\sigma_1^{\epsilon_1}\cdots d\sigma_n^{\epsilon_n}$. We define the *weight* of a polynomial $\psi(\sigma_1,\cdots,\sigma_n : d\sigma_1,\cdots, d\sigma_n)$ as the largest *weight* of its terms.

LEMMA 2.2. *$\{s(\alpha) \mid |\alpha| = t\}$ is a basis of $R_t^{\Sigma_n}$, the homogeneous polynomials of degree t invariant under the action of Σ_n.*

PROOF. If $\Sigma_{i=1}^l n_i s(\alpha_i) = 0$ where $n_i \in k$, then $n_i = 0$ for all $i = 1,\cdots,l$ since each monomials in $s(\alpha_i)$ are linearly independent and do not appear in any other $s(\alpha_j)$ for $j \neq i$. For any $f \in R_t^{\Sigma_n}$, we choose a monomial α_1 in f so that $s(\alpha_1) = \Sigma_{\sigma\in(\Sigma_n)_{\alpha_1}}\alpha_{1\sigma}$ is a homogeneous invariant form under Σ_n with degree t and also a summand of f. If $f = n_1 s(\alpha_1)$, then it is done. Otherwise we choose another monomial α_2 such that α_2 is not a summand of $s(\alpha_1)$. Similarly, $s(\alpha_2)$ is also an invariant summand of f with degree t. We repeat this for finite monomials. Therefore f is generated by $\{s(\alpha) \mid |\alpha| = t\}$. □

LEMMA 2.3. *Let f_1 be a symmetric polynomial in R. If f_1 vanishes for $x_n = y_n = 0$, then*

$$f_1 = \sum_{i=1}^m n_i\Big(\sum_{\sigma\in(\Sigma_n)_{\alpha_i}} \alpha_{i\sigma}\Big) + \sigma_n h(y_1,\cdots,y_n : x_1,\cdots,x_n)$$

where $n_i \in k$, $\alpha_i = y_1\cdots y_{n-1}x_n g_i(y_1,\cdots,y_{n-1} : x_1,\cdots,x_{n-1})$ and g_i is a monomial such that $y_n \nmid g_i$. We factor σ_n out of the terms which include σ_n from the second part of the right hand side.

PROOF. Since the symmetric polynomial f_1 is a linear combination of the basis $\{s(\gamma)\}$ where γ is some monomial in f_1, it is enough to find the proper invariant forms of $s(\gamma)$ under Σ_n. All terms of f_1 contain x_n or y_n because f_1 vanishes for $x_n = y_n = 0$. Thus we consider two cases to find the form of a symmetric polynomial f_1 which includes x_n or y_n in every term.

(I) First, we consider the terms containing x_n.

Let α be a monomial including x_n in f_1. We set

$$\alpha = x_n g(y_1, \cdots, y_n : x_1, \cdots, x_{n-1}) \qquad (*)$$

(Note $x_n^2 = 0$, thus we can assume $x_n \nmid g$.) We define $\alpha_\sigma = x_{\sigma(n)} g_\sigma(y_1, \cdots, y_n : x_1, \cdots, x_{n-1})$ for $\sigma \in (\Sigma_n)_\alpha$ where g_σ is g permuted by the action of σ. Since f_1 is symmetric, each term α_σ is also a summand of f_1 and $s(\alpha) = \sum_{\sigma \in (\Sigma_n)_\alpha} x_{\sigma(n)} g_\sigma$ is an invariant form in f_1 under Σ_n. Now we notice α has factors with subscripts $1, 2, \cdots, n$, because if we assume there is some i, $1 \leq i \leq n-1$, which does not occur as a subscript in g, then for $\sigma_i = (i, n) \in (\Sigma_n)_\alpha$, $\alpha_{\sigma_i} = x_i g_{\sigma_i}$ is also a summand of f_1. But g_{σ_i} does not have any factor with subscript n. This is a contradiction since all terms of f_1 include x_n or y_n. Thus $x_i \mid g$ if $y_i \nmid g$ for $i = 1, \cdots, n-1$. We wish to factor out all such x_i's, say $x_{i_1}, \cdots, x_{i_k}$ for some k. Thus we set

$$\alpha = x_n x_{i_1} \cdots x_{i_k} y_{i_{k+1}} \cdots y_{i_{n-1}} g_1(y_1, \cdots, y_n : x_1, \cdots, x_{n-1})$$

where $y_{i_j} \nmid g$, $1 \leq j \leq k$ and $\{i_1, \cdots, i_{n-1}\} = \{1, \cdots, n-1\}$.

Now we consider two cases.

(i) We assume g in $(*)$ does not include y_n as a factor.

If $k \geq 1$, for given $\sigma \in (\Sigma_n)_\alpha$, $\alpha_\sigma = x_{\sigma(n)} x_{\sigma(i_1)} \cdots x_{\sigma(i_k)} y_{\sigma(i_{k+1})} \cdots y_{\sigma(i_{n-1})} g_{1\sigma}$, where $y_{\sigma(i_j)} \nmid g_\sigma$, $1 \leq j \leq k$. Then there exists $\tau = (\sigma(n), \sigma(i_1)) \in (\Sigma_n)_\alpha$ such that $\sigma\tau = \sigma' \in (\Sigma)_\alpha$, and hence $\alpha_\sigma + \alpha_{\sigma'} = 0$ since $g_{1\sigma} = g_{1\sigma'}$. Here we understand $\alpha_{\sigma\tau} = (\alpha_\sigma)_\tau$. Each pair of terms of $s(\alpha)$ cancels, hence $s(\alpha) = 0$. Therefore $k = 0$ and in this case we get a proper monomial form,

$$\alpha = y_1 \cdots y_{n-1} x_n g_1(y_1, \cdots, y_{n-1} : x_1, \cdots, x_{n-1}). \qquad (*')$$

Therefore $s(\alpha) = \sum_{\sigma \in (\Sigma_n)_\alpha} y_{\sigma(1)} \cdots y_{\sigma(n-1)} x_{\sigma(n)} g_{1\sigma}$ is an invariant form in a symmetric polynomial f_1 which contains x_n or y_n in each term.

(ii) Next, we consider the case g in $(*)$ contains y_n as a factor, that is, α contains x_n and y_n. Then we set

$$\begin{aligned} \alpha &= x_n y_n x_{i_1} \cdots x_{i_k} y_{i_{k+1}} \cdots y_{i_{n-1}} g_1' \\ &= y_n x_{i_1} \cdots x_{i_k} y_{i_{k+1}} \cdots y_{i_{n-1}} g_2' \end{aligned}$$

where $y_{i_j} \nmid g$, $1 \leq j \leq k$ and $\{i_1, \cdots, i_{n-1}\} = \{1, \cdots, n-1\}$.

If $k \geq 2$, for given $\sigma \in (\Sigma_n)_\alpha$,
$\alpha_\sigma = y_{\sigma(n)} x_{\sigma(i_1)} \cdots x_{\sigma(i_k)} y_{\sigma(i_{k+1})} \cdots y_{\sigma(i_{n-1})} g'_{2\sigma}$, where $y_{\sigma(i_j)} \nmid g_\sigma$, $1 \leq j \leq k$. Then there exists $\rho = (\sigma(i_1), \sigma(i_2)) \in (\Sigma_n)_\alpha$ such that $\sigma\rho = \bar{\sigma} \in (\Sigma_n)_\alpha$, and hence $\alpha_\sigma + \alpha_{\bar{\sigma}} = 0$ since $g'_{2\sigma} = g'_{2\bar{\sigma}}$. Thus each pair of terms in $s(\alpha)$ cancels, i.e. $s(\alpha) = 0$. Therefore $k < 2$.

Case $k = 1$; We set $\alpha = y_n x_{i_1} y_{i_2} \cdots y_{i_{n-1}} g'_2$ where $y_{i_1} \nmid g$. Then we choose $\sigma_{i_1} = (i_1, n) \in (\Sigma_n)_\alpha$ such that

$$\begin{aligned} \alpha_{\sigma_{i_1}} &= x_n y_{i_1} y_{i_2} \cdots y_{i_{n-1}} g'_{2\sigma_{i_1}} \\ &= y_1 \cdots y_{n-1} x_n g_2 \end{aligned}$$

This $\alpha_{\sigma_{i_1}}$ is of the form $(*')$. Since $s(\alpha) = s(\alpha_{\sigma_{i_1}})$, we have the same invariant form of $s(\alpha)$ in (i) which includes x_n or y_n in every term.

Case $k = 0$; We set

$$\begin{aligned} \alpha &= y_n y_{i_1} \cdots y_{i_{n-1}} g'_2 \\ &= y_1 y_2 \cdots y_n g'_2 \\ &= \sigma_n g'_2 \end{aligned}$$

Therefore in this case we have $s(\alpha) = \sum_{\sigma \in (\Sigma_n)_\alpha} \sigma_n g'_{2\sigma} = \sigma_n h_1(y_1, \cdots, y_n : x_1, \cdots, x_n)$ where $h_1 = \sum_{\sigma \in (\Sigma_n)_\alpha} g'_{2\sigma}$ is a symmetric polynomial. This is another invariant form in f_1 which includes y_n in each term.

(II) Second, we consider the terms containing y_n.

Let β be a monomial including y_n in f_1. We set

$$\beta = y_n h'(y_1, \cdots, y_n : x_1, \cdots, x_n).$$

Since β has factors with subscripts $1, 2, \cdots, n$, we set

$$\beta = y_n x_{i_1} \cdots x_{i_k} y_{i_{k+1}} \cdots y_{i_{n-1}} h'_1(y_1, \cdots, y_n : x_1, \cdots, x_{n-1})$$

where $y_{i_j} \nmid h'$, $1 \leq j \leq k$ and $\{i_1, \cdots, i_{n-1}\} = \{1, \cdots, n-1\}$. If h' includes x_n as a factor, then this is the same case with (ii) in (I). Thus we assume h' does not include x_n as a factor.

Similarly if $k \geq 2$, for given $\sigma \in (\Sigma_n)_\beta$,
$\beta_\sigma = y_{\sigma(n)} x_{\sigma(i_1)} \cdots x_{\sigma(i_k)} y_{\sigma(i_{k+1})} \cdots y_{\sigma(i_{n-1})} h'_{1\sigma}$, where $y_{\sigma(i_j)} \nmid h'_\sigma$,

$1 \leq j \leq k$. Then there exists $\rho = (\sigma(i_1), \sigma(i_2)) \in (\Sigma_n)_\beta$ such that $\sigma\rho = \bar{\sigma} \in (\Sigma_n)_\beta$, and hence $\beta_\sigma + \beta_{\bar{\sigma}} = 0$ since $h'_{1\sigma} = h'_{1\bar{\sigma}}$. Each pair of terms in $s(\beta)$ cancels, hence $s(\beta) = 0$. Therefore $k < 2$.

Case $k = 1$; We set $\beta = y_n x_{i_1} y_{i_2} \cdots y_{i_{n-1}} h'_1$ where $y_{i_1} \nmid h'$. Then we choose $\sigma_{i_1} = (i_1, n) \in (\Sigma_n)_\beta$ such that

$$\begin{aligned}\beta_{\sigma_{i_1}} &= x_n y_{i_1} y_{i_2} \cdots y_{i_{n-1}} h'_{1\sigma_{i_1}} \\ &= y_1 \cdots y_{n-1} x_n g_3\end{aligned}$$

Now $\beta_{\sigma_{i_1}}$ is of the form $(*')$ and since $s(\beta) = s(\beta_{\sigma_{i_1}})$, $s(\beta)$ has the same invariant form in (i) which contains x_n or y_n in each term.

If $k = 0$,

$$\begin{aligned}\beta &= y_n y_{i_1} \cdots y_{i_{n-1}} h'_1 \\ &= y_1 y_2 \cdots y_n h'_1 \\ &= \sigma_n h'_1\end{aligned}$$

Then we have $s(\beta) = \sum_{\sigma \in (\Sigma_n)_\beta} \sigma_n h'_{1\sigma} = \sigma_n h_2(y_1, \cdots, y_n : x_1, \cdots, x_n)$ where $h_2 = \sum_{\sigma \in (\Sigma_n)_\beta} h'_{1\sigma}$ is a symmetric polynomial. Thus $s(\beta)$ is an invariant form in f_1 which includes y_n in every term.

Therefore f_1 is the linear combination of the invariant forms we obtained through (I) and (II). □

THEOREM 2.4.

$$\begin{aligned}&(k[y_1, \cdots, y_n : 2] \otimes E[x_1, \cdots, x_n : 1])^{\Sigma_n} \\ &= k[\sigma_1, \cdots, \sigma_n] \otimes E[d\sigma_1, \cdots, d\sigma_n]\end{aligned}$$

PROOF. Since each σ_i and $d\sigma_i$ are symmetric functions, clearly $k[\sigma_1, \cdots, \sigma_n] \otimes E[d\sigma_1, \cdots, d\sigma_n] \subset (k[y_1, \cdots, y_n : 2] \otimes E[x_1, \cdots, x_n : 1])^{\Sigma_n}$. We need to show that the converse is also true, i.e. a symmetric polynomial of degree l may be written as a polynomial $\psi(\sigma_1, \cdots, \sigma_n : d\sigma_1, \cdots, d\sigma_n)$ of weight l. The proof is given by induction on n. For $n = 1$, $(k[y_1] \otimes E[x_1])^{\Sigma_1} = k[y_1] \otimes E[x_1]$. Here $y_1 = \sigma_1$ and $x_1 = d\sigma_1$, hence $(k[y_1] \otimes E[x_1])^{\Sigma_1} \subset k[\sigma_1] \otimes E[d\sigma_1]$. We assume it is true for $\leq n-1$.

To show it is true for n, we use induction on the degree of f again. For the polynomials of degree zero, it is trivial. We also assume it holds for symmetric polynomials with degree $\leq l-1$. Now let f be a symmetric polynomial of degree l. If we take $x_n = y_n = 0$, then $f(y_1, \cdots, y_{n-1}, 0 : x_1, \cdots, x_{n-1}, 0) = \varphi((\sigma_1)_0, \cdots, (\sigma_{n-1})_0 : (d\sigma_1)_0, \cdots, (d\sigma_{n-1})_0)$ by the induction, where $(\sigma_i)_0, (d\sigma_i)_0$ are generators of the symmetric functions with $n-1$ variables and φ is a symmetric function of weight $\leq l$. Thus the function $\varphi(\sigma_1, \cdots, \sigma_{n-1} : d\sigma_1, \cdots, d\sigma_{n-1})$ has weight $\leq l$.

Now we form $f_1 = f(y_1, \cdots, y_n : x_1, \cdots, x_n) - \varphi(\sigma_1, \cdots, \sigma_{n-1} : d\sigma_1, \cdots, d\sigma_{n-1})$. The polynomial f_1 is obviously symmetric and *deg* $f_1 \leq l$. Furthermore, for $x_n = y_n = 0$, f_1 vanishes. Therefore all terms of f_1 contain x_n or y_n. By Lemma 2.3, we set $f_1 = \sum_{i=1}^{m}(\sum_{\sigma\in(\Sigma_n)_{\alpha_i}} \alpha_{i\sigma}) + \sigma_n h(y_1, \cdots, y_n : x_1, \cdots, x_n)$ where $\alpha_i = y_1 \cdots y_{n-1} x_n g_i(y_1, \cdots, y_{n-1} : x_1, \cdots, x_{n-1})$ and g_i is a monomial such that $y_n \nmid g_i$. If we set $y_n = 0$ in the symmetric polynomial f_1, then every term of the form $y_1 \cdots \widehat{y}_j \cdots y_n x_j$, $j = 1, 2, \cdots, n-1$, vanishes. Therefore

$$
\begin{aligned}
f_1 &= \sum_{i=1}^{m}\Big(\sum_{\sigma\in\Sigma_{n-1}} y_1 \cdots y_{n-1} x_n g_{i\sigma}\Big) \\
&= (y_1 \cdots y_{n-1} x_n)\Big(\sum_{i=1}^{m} \sum_{\sigma\in\Sigma_{n-1}} g_{i\sigma}\Big)
\end{aligned}
$$

If we set $g_0 = \sum_{i=1}^{m} \sum_{\sigma\in\Sigma_{n-1}} g_{i\sigma}$, then since g_0 is symmetric, by the induction $g_0(y_1, \cdots, y_{n-1} : x_1, \cdots, x_{n-1}) = g((\sigma_1)_0, \cdots, (\sigma_{n-1})_0 : (d\sigma_1)_0, \cdots, (d\sigma_{n-1})_0)$. Thus if $y_n = 0$, we have
$f_1 = (y_1 \cdots y_{n-1} x_n) g((\sigma_1)_0, \cdots, (\sigma_{n-1})_0 : (d\sigma_1)_0, \cdots, (d\sigma_{n-1})_0)$. The weight of $g((\sigma_1)_0, \cdots, (\sigma_{n-1})_0 : (d\sigma_1)_0, \cdots, (d\sigma_{n-1})_0)$ is $\leq l-2n+1$. Thus the function of $g(\sigma_1, \cdots, \sigma_{n-1} : d\sigma_1, \cdots, d\sigma_{n-1})$ has weight $\leq l-2n+1$. Now we form $f_2 = f_1 - (d\sigma_n) g(\sigma_1, \cdots, \sigma_{n-1} : d\sigma_1, \cdots, d\sigma_{n-1})$. f_2 is also a symmetric polynomial of degree $\leq l$ and f_2 vanishes if $y_n = 0$. (Note $x_n(d\sigma_i)|_{y_n=0} = x_n(d\sigma_i)_0$, $1 \leq i \leq n-1$). Therefore all terms of f_2 contain y_n. Since f_2 is symmetric, all terms also include the factors $y_1, \cdots, y_{n-1}$. Factoring the product $\sigma_n = y_1 \cdots y_n$ out of all terms, we obtain $f_2 = \sigma_n h_0(y_1, \cdots, y_n : x_1, \cdots, x_n)$ where h_0 is again a symmetric polynomial of degree $\leq l-2n < l$. By induction on l, $h_0(y_1, \cdots, y_n : x_1, \cdots, x_n) = h(\sigma_1, \cdots, \sigma_n : d\sigma_1, \cdots, d\sigma_n)$

and $f_2 = \sigma_n h(\sigma_1, \cdots, \sigma_n : d\sigma_1, \cdots, d\sigma_n)$. Therefore we have $f_1 = (d\sigma_n) g(\sigma_1, \cdots, \sigma_{n-1} : d\sigma_1, \cdots, d\sigma_{n-1}) + \sigma_n h(\sigma_1, \cdots, \sigma_n : d\sigma_1, \cdots, d\sigma_n)$.

Hence we represent f as

$$\begin{aligned} f &= f_1 + \varphi(\sigma_1, \cdots, \sigma_{n-1} : d\sigma_1, \cdots, d\sigma_{n-1}) \\ &= (d\sigma_n) g(\sigma_1, \cdots, \sigma_{n-1} : d\sigma_1, \cdots, d\sigma_{n-1}) + \sigma_n h(\sigma_1, \cdots, \sigma_n : \\ &\qquad d\sigma_1, \cdots, d\sigma_n) + \varphi(\sigma_1, \cdots, \sigma_{n-1} : d\sigma_1, \cdots, d\sigma_{n-1}). \end{aligned}$$

The polynomial of the right hand side has weight at most l. But the weight can not be less than l, since, otherwise f would be of degree $< l$. Therefore the right side is exactly of weight l. Now we conclude that a symmetric polynomial of degree l may be expressed as a polynomial $\psi(\sigma_1, \cdots, \sigma_n : d\sigma_1, \cdots, d\sigma_n)$ of weight l. □

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