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NOTE ON NONPATH-CONNECTED ORTHOMODULAR LATTICES

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ABSTRACT. Some nonpath-connectecd orthomodular lattices are given: Every infinite direct product of othomodular lattices containing infinitely many non-Boolean factors is a nonpath-connected orthomodular lattice. The orthomodular lattice of all closed subspaces of an infinite dimensional Hilbert space is a nonpath-connected orthomodular lattice.

1. Preliminaries

An *orthomodular lattice* (abbreviated by OML) is an ortholattice L which satisfies *the orthomodular law*: if $x \leq y$, then $y = x \vee (x' \wedge y)$ [5]. A *Boolean algebra* B is an ortholattice satisfying the *distributive law*: $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \quad \forall x, y, z \in B$.

A *subalgebra* of an OML L is a nonempty subset M of L which is closed under the operations $\vee$, $\wedge$ and $'$. We write $M \leq L$ if M is a subalgebra of L. If $M \leq L$ and $a, b \in M$ with $a \leq b$, then the *relative interval sublattice* $M[a,b] = \{x \in M \mid a \leq x \leq b\}$ is an OML with the *relative orthocomplementation* $^\sharp$ on $M[a,b]$ given by $c^\sharp = (a \vee c') \wedge b = a \vee (c' \wedge b) \quad \forall c \in M[a,b]$. In particular, $L[a,b]$ will be denoted by $[a,b]$ if there is no ambiguity.

The *commutator of a and b* of an OML L is denoted by $a * b$, and is defined by $a * b = (a \vee b) \wedge (a \vee b') \wedge (a' \vee b) \wedge (a' \vee b')$. The set of all commutators of L is denoted by $ComL$ and L is said to be *commutator-finite* if $|ComL|$ is finite. For elements a, b of an OML, we say *a commutes with b*, in symbols $a\,\mathbf{C}\,b$, if $a * b = 0$. If M is a subset of an OML L, the set $\mathbf{C}(M) = \{x \in L \mid x\,\mathbf{C}\,m \quad \forall m \in M\}$ is called the *commutant* of M in L and the set $\mathbf{C}en(M) = \mathbf{C}(M) \cap M$ is called the *center* of M. The

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set $\mathbf{C}(L)$ is called the center of L and then $\mathbf{C}(L) = \bigcap \{\mathbf{C}(a) \mid a \in L\}$. An OML L is called *irreducible* if $\mathbf{C}(L) = \{0, 1\}$, and L is called *reducible* if it is not irreducible.

A *block* of an OML L is a maximal Boolean subalgebra of L. The set of all blocks of L is denoted by $\mathcal{A}_L$. Note that $\bigcup \mathcal{A}_L = L$ and $\bigcap \mathcal{A}_L = \mathbf{C}(L)$. An OML L is said to be *block-finite* if $|\mathcal{A}_L|$ is finite.

For any e in an OML L, the subalgebra $S_e = [0, e'] \cup [e, 1]$ is called the *(principal) section generated by e*. Note that for $A, B \in \mathcal{A}_L$, if $e \in (A \cap B)$ and $A \cap B = S_e \cap (A \cup B)$, then $A \cap B = S_e \cap A = S_e \cap B$.

DEFINITION 1.1. For blocks A, B of an OML L define $A \overset{wk}{\sim} B$ if and only if $A \cap B = S_e \cap (A \cup B)$ for some $e \in A \cap B$; $A \sim B$ if and only if $A \neq B$ and $A \cup B \leq L$; $A \approx B$ if and only if $A \sim B$ and $A \cap B \neq \mathbf{C}(L)$.

A *path* in L is a finite sequence B_0, B_1, ..., B_n $(n \geq 0)$ in $\mathcal{A}_L$ satisfying $B_i \sim B_{i+1}$ whenever $0 \leq i < n$. The path is said to *join* the blocks B_0 and B_n. The number n is said to be the *length* of the path. A path is said to be *proper* if and only if $n = 1$ or $B_i \approx B_{i+1}$ holds whenever $0 \leq i < n$. A path is called to be *strictly proper* if and only if $B_i \approx B_{i+1}$ holds whenever $0 \leq i < n$ [1].

Note that $A \approx B$ implies $A \sim B$, and $A \sim B$ implies $A \overset{wk}{\sim} B$. Some authors, for example Greechie, use the phrase "*A and B meet in the section S_e*" to describe $A \overset{wk}{\sim} B$ [3].

DEFINITION 1.2. Let L be an OML, and $A, B \in \mathcal{A}_L$. We will say that A and B are *path-connected in L*, *strictly path-connected in L* if A and B are joined by a proper path, a strictly proper path, respectively. We will say A and B are *nonpath-connected* if there is no proper path joining A and B, and L is called *nonpath-connected* if there exist two blocks which are nonpath-connected. An OML L is called *path-connected in L*, *strictly path-connected in L* if any two blocks in L are joined by a proper path, a strictly proper path, respectively. An OML L is called *relatively path-connected* iff each $[0, x]$ is path-connected for all $x \in L$.

The following two lemmas are well known.

LEMMA 1.3. *If L is an OML with two blocks A, B and $a \in A \setminus B$ and $b \in B \setminus A$, then $A \cap B = S_{a*b}$. If $A \cap B = S_c$, then $c = a * b$ [1].*

Let A, B be two blocks of an OML L. If $A \sim B$ holds, then there exists a unique element $e \in A \cap B$ satisfying $A \cap B = (A \cup B) \cap S_e$ by lemma 2.2. Therefore we say that A and B are *linked at* e (*strongly linked at* e) if $A \sim B$ ($A \approx B$) and use the notation $A \sim_e B$ ($A \approx_e B$).

LEMMA 1.4. [Bruns] *If L_1, L_2 are OMLs, $L = L_1 \times L_2$, $A, B \in \mathcal{A}_{L_1}$ and $C, D \in \mathcal{A}_{L_2}$, then $A \times C \sim B \times D$ holds in L if and only if either $A = B$ and $C \sim D$ or $A \sim B$ and $C = D$. If A and B are linked at a then $A \times C$ and $B \times C$ are linked at $(a, 0)$. If C and D are linked at c then $A \times C$ and $A \times D$ are linked at $(0, c)$ [1].*

An OML L is called the *horizontal sum* of a family $(L_i)_{i \in I}$ (denoted by $\circ(L_i)_{i \in I}$) of at least two subalgebras, if $\bigcup L_i = L$, and $L_i \cap L_j = \{0, 1\}$ whenever $i \neq j$, and one of the following equivalent conditions is satisfied:
(1) if $x \in L_i \setminus L_j$ and $y \in L_j \setminus L_i$, then $x \vee y = 1$;
(2) every block of L belongs to some L_i;
(3) if S_i is a subalgebra of L_i, then $\bigcup S_i$ is a subalgebra of L [2].

Note that the horizontal sum of a family $(L_i)_{i \in I}$ of path-connected OML L_i ($i \in I$) is a path-connected OML.

Bruns introduced a construction which is more general than the horizontal sum, and proved the following lemma 1.5 [1].

An OML L is said to be the *weak horizontal sum* of a family $(L_i)_{i \in I}$ of subalgebras if and only if there exists an isomorphism f of L onto a product of $L_0 \times L'$ of a Boolean algebra L_0 and an OML L' such that the subalgebra L_i of L correspond via f to subalgebras of the form $L_0 \times L'_i$ and L' is the the horizontal sum of the family $(L'_i)_{i \in I}$.

LEMMA 1.5. *Every OML L with only two blocks is isomorphic with an OML of the form $\mathbf{B} \times (\mathbf{A} \circ \mathbf{C})$ where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are Boolean algebras and $\mathbf{A} \circ \mathbf{C}$ is the horizontal sum of $\mathbf{A}$ and $\mathbf{C}$. In other words, every OML with only two blocks is the weak horizontal sum of its blocks [1].*

2. Non path-connected orthomodular lattices

We know that every finite direct product of path-connected OMLs is path-connected [8], and we have the following class of nonpath-connected OMLs. We can find some examples and properties of path-connected OMLs in [1, 2, 7, 8].

PROPOSITION 2.1. *Every infinite direct product of OMLs containing infinitely many non-Boolean factors is a nonpath-connected OML.*

PROOF. Let $L \cong \prod_{\alpha \in I} L_\alpha$ where $|I| \geq \omega$ and each L_α is OML. Let $J = \{j \in I \mid L_j$ is a Boolean algebra $\}$. Then $\mathbf{B} = \prod_{j \in J} L_j$ is a Boolean factor of L. Thus $L \cong \mathbf{B} \times \prod_{i \in I \setminus J} L_i$ such that $|I \setminus J| \geq \omega$ and each L_i $(i \in I \setminus J)$ is non Boolean path-connected OML. Therefore it is sufficient to show that $\prod_{i \in I \setminus J} L_i$ is not path-connected by lemma 1.4. Since each L_i $(i \in I \setminus J)$ is a path-connected OML containing at least two distinct blocks, there exist distinct $A_i, B_i \in \mathcal{A}_{L_i}$ $\forall i \in I \setminus J$ such that $A_i \cup B_i \leq L_i$. Let $A = \prod_{i \in (I \setminus J)} A_i$ and $B = \prod_{i \in (I \setminus J)} B_i$. Then A and B are not path-connected since there is no path of finite length from A to B by lemma 1.4. ∎

For the remainder of this paper, let H be an infinite dimensional Hilbert space over the real or complex numbers. A *linear manifold* is a nonempty subset M of H such that if x and y are in M, then $ax + by \in M$ for every pair of complex numbers a and b. A *closed subspace* is a closed linear manifold. The *closed subspace spanned by* an arbitrary subset M of H is defined to be the intersection of all closed subspaces containing M. The *vector sum* of two closed subspaces M and N, in symbols $M + N$, is defined to be the set of all vectors of the form $x + y$ with $x \in M$ and $y \in N$. If M and N are closed subspaces, we use the symbol $M \vee N$ for the closed subspace spanned by M and N. It follows by this definition that $M \vee N$ is the smallest closed subspace containing both M and N.

We need the following lemmas to prove that the OML $\mathcal{C}(H)$ of all closed subspaces of an infinite dimensional Hilbert space H is a nonpath-connected OML.

LEMMA 2.2. *Let A, B be distinct atomic blocks of $\mathcal{C}(H)$ with $A \cup B \leq L$. Then there exists $\alpha \in Com\,\mathcal{C}(H)$ such that $A \cup B = (A \cup B)[0, \alpha'] \oplus (A \cup B)[0, \alpha]$ where $(A \cup B)[0, \alpha] \cong MO2$ and $(A \cup B)[0, \alpha']$ is a Boolean algebra. In particular, $h(\alpha) = 2$ in $\mathcal{C}(H)$.*

PROOF. We know that $A \cup B = (A \cup B)[0, \alpha'] \oplus (A \cup B)[0, \alpha]$ for some $\alpha \in Com\ \mathcal{C}(H)$ by lemma 1.3 where $(A \cup B)[0, \alpha]$ is a horizontal sum of $A[0, \alpha]$ and $B[0, \alpha]$, and $(A \cup B)[0, \alpha']$ is a Boolean algebra by lemma 1.5. Therefore it is sufficient to show that $h(\alpha) = 2$. Let a, b be two distinct atoms of $\mathcal{C}(H)$ such that $a \in A \setminus B$ and $b \in B \setminus A$. Then $\alpha = a \vee b$ since $(A \cup B)[0, \alpha]$ is a horizontal sum of $A[0, \alpha]$ and $B[0, \alpha]$. Thus $h(\alpha) = h(a \vee b) = 2$ since a, b are atoms. We are done. ∎

LEMMA 2.3. *If F is a finite dimensional linear manifold in a Hilbert space H, and if S is a closed subspace in H, then the vector sum $F + S$ is necessarily closed (and hence is therefore equal to the span $F \vee S$) [p9, 6].*

As a consequence of lemma 2.3 every finite dimensional linear manifold is closed, since $S = \{0\}$ is a closed subspace of H.

LEMMA 2.4. *If F is a finite dimensional closed subpace of a Hilbert space H and S is a closed subspace of H such that $S^{\perp}$ is an infinite dimensional closed subspace of H. Then the closed subspace $F \vee S$ in $\mathcal{C}(H)$ spanned by F and S is a proper closed subspace of H.*

PROOF. The closed subspace $F \vee S$ of H spanned by F and S is equal to $F + S$ by lemma 2.3. Thus the quotient space $(F \vee S)/S = (F + S)/S$ is a proper closed subspace in $S^{\perp}$ since F is finite dimensional and $S^{\perp}$ is infinite dimensional. Hence $S \vee F$ is a proper closed subspace of H. ∎

LEMMA 2.5. *If B is a block of $\mathcal{A}_{\mathcal{C}(H)}$, then there exists a unique element $x \in \mathcal{C}(H)$ such that $B = B[0, x] \oplus B[0, x']$ where $B[0, x]$ is atomic and $B[0, x']$ totally nonatomic. Moreover, B is atomic iff $x = 1$; and B is totally nonatomic iff $x = 0$.*

PROOF. Let $\{a_i\}_{i \in I}$ be the set of all atoms in a block B of the OML $\mathcal{C}(H)$. Then $\bigvee_{i \in I} a_i$ is exists since $\mathcal{C}(H)$ is complete [p65, 5]. Let $x = \bigvee_{i \in I} a_i$. Then $x \in B$ since B is subcomplete, $B[0, x]$ is atomic and $B[0, x']$ is totally nonatomic. ∎

LEMMA 2.6. *Let A be an atomic block of $\mathcal{C}(H)$, and B be a nonatomic block of $\mathcal{C}(H)$. Then $A \cup B \not\leq \mathcal{C}(H)$.*

PROOF. Suppose $A \cup B \leq \mathcal{C}(H)$. Then $A \cup B = (A \cup B)[0, \alpha] \oplus (A \cup B)[0, \alpha']$ for some $\alpha \in Com\ \mathcal{C}(H)$ by lemma 1.3 where $(A \cup B)[0, \alpha]$ is a horizontal sum of $A[0, \alpha]$ and $B[0, \alpha]$, and $(A \cup B)[0, \alpha']$ is a Boolean algebra by lemma (1.5). Moreover, $(A \cup B)[0, \alpha']$ is atomic since $A[0, \alpha']$

is atomic and $(A \cup B)[0, \alpha'] = A[0, \alpha']$. By lemma 2.5, there exists $x \in B$ such that $B[0, x]$ is totally nonatomic. Since $A[0, \alpha']$ is atomic, $B[0, x \wedge \alpha']$ is totally nonatomic and $x \wedge \alpha' \in B[0, \alpha'] = A[0, \alpha']$ it follows that $x \wedge \alpha' = 0$. Thus $x = (x \wedge \alpha) \vee (x \wedge \alpha') = x \wedge \alpha$ so that $x \leq \alpha$. We may assume that $0 < x < \alpha$. Moreover $h(x) = \infty$ since $[0, x]$ is nonatomic. Let a be an atom of $A[0, \alpha]$. Then $a \vee x = a \vee (x' \wedge \alpha) = \alpha$, since $(A \cup B)[0, \alpha]$ is a horizontal sum of $A[0, \alpha]$ and $B[0, \alpha]$ and $x' \wedge \alpha$ is the relative orthocomplement of X in $(A \cup B)[0, x]$. This contradicts $a \vee (x' \wedge \alpha) < \alpha$ by applying lemma 2.4 to the Hilbert space α since $h(a) = 1$, $h(\alpha) = \infty$, x is the orthocomplement of $x' \wedge \alpha$ in $B[0, \alpha]$, and $h(x) = \infty$. We are done. ∎

LEMMA 2.7. *Let A and B be atomic path-connected blocks of a $\mathcal{C}(H)$ with a path $A = C_0 \sim C_1 \sim C_2 \sim \ldots \sim C_{(n-1)} \sim C_n = B$. Then $A \cap B \supseteq S_x \cap (A \cup B)$ for some $x \in (A \cap B)$ with $h(x) \leq 2n$.*

PROOF. We will prove the conclusion by induction on the length k of the path joining atomic blocks A and B of a $\mathcal{C}(H)$. If $k = 1$, then $A = C_0 \sim C_1 = B$. Thus $A \cap B = S_x \cap (A \cup B)$ for some $x \in (A \cap B)$ where $h(x) = 2$ by lemma 2.2. Assume that the conclusion of the lemma is true for each path joining two blocks of $\mathcal{C}(H)$ with the length less than or equal to $n - 1$. Let $A = C_0 \sim C_1 \sim \ldots \sim C_{n-1} \sim C_n = B$ be a path from A to B of length n. We may assume that $C_{n-1} \neq A$ otherwise, $A = C_{n-1} \sim B$ we are done by the case $k = 1$. By induction hypothesis, $(A \cap C_{n-1}) \supseteq S_x \cap (A \cup C_{n-1})$ for some $x \in (A \cap C_{n-1})$ with $h(x) \leq 2(n-1)$, and $(C_{n-1} \cap B) \supseteq S_y \cap (C_{n-1} \cup B)$ for some $y \in (C_{n-1} \cap B)$ with $h(y) \leq 2$. Thus $(A \cap B) \supseteq (A \cap C_{n-1}) \cap (C_{n-1} \cap B) \supseteq (S_x \cap (A \cup C_{n-1})) \cap (S_y \cap (C_{n-1} \cup B))$, and $(S_x \cap (A \cup C_{n-1})) \cap (S_y \cap (C_{n-1} \cup B)) \supseteq S_{(x \vee y)} \cap (A \cup B)$ since $S_x \cap (A \cup C_{n-1}) = S_x \cap A = S_x \cap C_{n-1}$, $S_y \cap (C_{n-1} \cup B) = S_y \cap C_{n-1} = S_y \cap B$ and $S_x \cap S_y \supseteq S_{(x \vee y)}$. Moreover, $x \vee y \in (A \cap B)$ and $h(x \vee y) \leq h(x) + h(y) \leq 2(n-1) + 2 = 2n$. We are done. ∎

LEMMA 2.8. [Greechie] *Let L be an OML, let $\{e_\alpha \mid \alpha \in I\}$ be a maximal orthogonal family of nonzero elements of L, let $\{B_\alpha \mid \alpha \in I\}$ be a collection of atomic blocks of L such that $e_\alpha \in B_\alpha$ for all $\alpha \in I$, let $M = \bigcup\{B_\alpha[0, e_\alpha] \mid \alpha \in I\}$, and let $B = \mathbf{C}(M)$. Then B is an atomic block of L [3].*

Now, we are ready to prove one of our main theorems.

THEOREM 2.9. *The OML $\mathcal{C}(H)$ of all closed subspaces of an infinite dimensional Hilbert space H is a nonpath-connected OML.*

PROOF. First, assume that H is a separable Hilbert space. Let $(e_1, e_2, e_3, ...)$ be an orthonormal basis of the separable Hilbert space H. Let $f_{2i-1} = \dfrac{e_{2i-1} + e_{2i}}{\sqrt{2}}$ and $f_{2i} = \dfrac{e_{2i-1} - e_{2i}}{\sqrt{2}}$ $\forall(1 \leq i < \infty)$. Then $(f_1, f_2, f_3, ...)$ is an orthonormal basis of H. Let $[e_i] = \{\lambda e_i \mid \lambda \in \mathcal{C}\}$ and $[f_i] = \{\lambda f_i \mid \lambda \in \mathcal{C}\}$ $\forall 1 \leq i$ where $\mathcal{C}$ is the complex numbers, let $A = \mathbf{C}(\{[e_i] \mid 1 \leq i\})$, and let $B = \mathbf{C}(\{[f_i] \mid 1 \leq i\})$. Then A and B are atomic blocks in $\mathcal{C}(H)$ by lemma 2.8 since $\{[e_i] \mid i \in I\}$ and $\{[f_i] \mid i \in I\}$ are maximal orthogonal families of atoms of $\mathcal{C}(H)$. We claim that A and B are nonpath-connected in $\mathcal{C}(H)$. Suppose A and B are path-connected with a path $A = C_0 \sim C_1 \sim, ..., \sim C_{n-1} \sim C_n = B$. $A \neq B$ and $A \not\sim B$, since $A \cup B \not\leq \mathcal{C}(H)$ by lemma 2.2. Thus $n \geq 2$. If the path joining A and B contains only atomic blocks, then by lemma 2.7 $A \cap B \supseteq S_x \cap (A \cup B)$ for some $x \in A \cap B$ with $h(x) \leq 2n$, contradicting there is no $x \in A \cap B$ such that $A \cap B \supseteq S_x \cap (A \cup B)$ with $h(x) \leq 2n$ by our choice of A and B. Thus we may assume that one of $C_1, C_2, ..., C_{n-1}$ is nonatomic. Let C_i be the nonatomic block with the smallest index in the path joining A and B, and hence C_{i-1} is atomic. Thus $C_{i-1} \sim C_i$, and hence $C_{i-1} \cup C_i \leq \mathcal{C}(H)$ contradicting $C_{i-1} \cup C_i \not\leq \mathcal{C}(H)$ by lemma 2.6. Therefore A and B are nonpath-connected.

Finally, if H is nonseparable infinite dimensional Hilbert space, then there exists $x \in \mathcal{C}(H)$ such that x is a separable infinite dimensional Hilbert subspace of H. Let $(g_1, g_2, g_3, ...)$ be an orthonormal basis of x. Let $h_{2i-1} = \dfrac{g_{2i-1} + g_{2i}}{\sqrt{2}}$ and $h_{2i} = \dfrac{g_{2i-1} - g_{2i}}{\sqrt{2}}$ $\forall(1 \leq i < \infty)$. Then $(h_1, h_2, h_3, ...)$ is an orthonormal basis of x. Let $[g_i] = \{\lambda g_i \mid \lambda \in \mathcal{C}\}$ and $[h_i] = \{\lambda g_i \mid \lambda \in \mathcal{C}\}$ $\forall 1 \leq i$ where $\mathcal{C}$ is the set of all complex numbers, let $D = \mathbf{C}(\{[g_i] \mid 1 \leq i\})$, and let $E = \mathbf{C}(\{[h_i] \mid 1 \leq i\})$. Then D and E are an atomic blocks in x by the above argument. Let F be an atomic block of x'. Then $D \oplus F$ and $E \oplus F$ are distinct atomic blocks of $\mathcal{C}(H)$. Now, the desired conclusion follows by applying lemmas 2.2, 2.7 and 2.6 to the blocks $D \oplus F$ and $E \oplus F$. ∎

Corollary 2.10. *H is a finite dimensional Hilbert space if and only if $C(H)$ is path-connected.*

Proof. If H is finite dimensional, then $C(H)$ is chain-finite. Thus $C(H)$ is path-connected since every chain-finite OML is path-connected[7]. Conversely, if H is infinite dimensional, then $C(H)$ is nonpath-connected by theorem 2.9. ∎

References

1. Bruns, G., *Block-finite Orthomodular Lattices*, Can. J. Math. **31** (1979), 961-985.
2. Bruns, G. and Greechie, R., *Blocks and Commutators in Orthomodular Lattices*, Algebra Universalis **27** (1990), 1-9.
3. Greechie, R., *On the Structure of Orthomodular Lattices Satisfying the Chain Condition*, J. of Combinatorial Theory **4** (1968), 210-218.
4. Greechie, R. and Herman, L., *Commutator-finite Orthomodular Lattices*, Order **1** (1985), 277-284.
5. Kalmbach, G., *Orthomodular Lattices*, Academic Press Inc (London) Ltd., 1983.
6. Halmos, P., *A Hilbert Space Problem Book*, Springer-Verlag, New York, 1983.
7. Park, E., *Path-connected Orthomodular Lattices*, Kansas State University Ph. D. Thesis, 1989.
8. Park, E., Bull. of Korean Math. Soc. **31** (1994), 61-72.

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