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ON $\wedge$-ASSOCIATIVE BCI-ALGEBRAS

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ABSTRACT. In this paper, we introduce the notion of $\wedge$-associative BCI-algebras, investigate important properties of such algebras, and discuss the structure of it.

Meng and Xin [4] introduced the notion of commutative BCI-algebras as a generalization of one of commutative BCK-algebras. Chaudhry [1] and Hoo [2] also introduced the notions called pseudo-commutative BCI-algebras and branchwise commutative BCI-algebras. In this paper, we introduce the concept of $\wedge$-associative BCI-algebras, which is also a generalization of the notion of commutative BCK-algebras. We investigates some properties of such algebras. We show that every $\wedge$-associative BCI-algebra is commutative, and give an example showing that the converse is not true. We also give a characterization of $\wedge$-associative BCI-algebras.

Let us recall some definitions and results which are necessary for development of this paper.

By a BCI-algebra we mean an algebra $(X; *, 0)$ of type $(2, 0)$ satisfying the following axioms:

(I) $((x * y) * (x * z)) * (z * y) = 0$,

(II) $(x * (x * y)) * y = 0$,

(III) $x * x = 0$,

(IV) $x * y = 0$ and $y * x = 0$ imply that $x = y$,

for all $x, y, z \in X$. A partial ordering $\leq$ on X can be defined by $x \leq y$ if and only if $x * y = 0$.

A BCI-algebra X satisfying (V) $0 * x = 0$ for all $x \in X$ is called a BCK-algebra.

In a BCI-algebra X, the following hold:

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(1) $x * 0 = x$,
(2) $(x * y) * z = (x * z) * y$,
(3) $x * (x * (x * y)) = x * y$,
(4) $0 * (x * y) = (0 * x) * (0 * y)$,
(5) $((x * z) * (y * z)) * (x * y) = 0$,
(6) $x \leq y$ implies $x * z \leq y * z$ and $z * y \leq z * x$.

For any BCI-algebra X, the BCK-part of X is the set

$$B(X) = \{x \in X | 0 \leq x\}.$$

If $B(X) = \{0\}$, we say that X is a p-semisimple BCI-algebra.

Let X be a BCI/BCK-algebra. Denote $y * (y * x)$ by $x \wedge y$ for all $x, y \in X$. We note that $x \wedge y \leq x, y$ for every x and y in a BCK/BCI-algebra X.

PROPOSITION 1. *Let X be a BCK-algebra. Then the following are equivalent:*
(7) $x \wedge y = y \wedge x$.
(8) $x \leq y$ *implies* $x = x \wedge y$.
(9) $(x \wedge y) \wedge x = x \wedge y$.
(10) $(x \wedge y) \wedge z = x \wedge (y \wedge z)$.

PROOF. From [4; Theorems 1 and 3] it follows immediately that (7), (8) and (9) are equivalent. Noticing that a BCK-algebra X is commutative if and only if $(X, \wedge)$ is a semilattice, it is clear that (7) and (10) are equivalent.

A BCK-algebra X is said to be commutative if it satisfies (7). A BCI-algebra X is said to be commutative if it satisfies (8).

DEFINITION 2. A BCI-algebra X is said to be $\wedge$-associative if it satisfies (10).

EXAMPLE 3. (i) A commutative BCK-algebra is a $\wedge$-associative BCI-algebra.

(ii) A p-semisimple BCI-algebra is $\wedge$-associative.

(iii) Let X be the set $\{0, 1, a\}$ with Cayley table as follows:

*	0	1	a
0	0	0	a
1	1	0	a
a	a	a	0

Then it is easy to verify that X is a $\wedge$-associative BCI-algebra. But X is neither a commutative BCK-algebra nor a p-semisimple BCI-algebra.

THEOREM 4. *If a BCI-algebra X is $\wedge$-associative then it is commutative. But the converse does not hold.*

PROOF. Suppose that X is a $\wedge$-associative BCI-algebra. Let $x, y \in X$ be such that $x \leq y$. Then $x = x \wedge x = x \wedge (x * 0) = x \wedge (x * (x * y)) = x \wedge (y \wedge x) = (x \wedge y) \wedge x \leq x \wedge y \leq x$, hence $x = x \wedge y$. Therefore X is commutative. The following example shows that the converse does not hold.

EXAMPLE 5. Let X be the set $\{0, 1, 2, a, b, c, d\}$ with Cayley table as follows:

*	0	1	2	a	b	c	d
0	0	0	0	a	a	a	a
1	1	0	0	b	a	a	a
2	2	1	0	c	b	a	a
a	a	a	a	0	0	0	0
b	b	a	a	1	0	0	0
c	c	b	a	2	1	0	1
d	d	b	a	2	1	1	0

By routine calculations we know that X is a commutative BCI-algebra. But X is not $\wedge$-associative because $(2 \wedge c) \wedge d = (c * (c * 2)) \wedge d = 2 \wedge d = d * (d * 2) = 2$ and $2 \wedge (c \wedge d) = 2 \wedge (d * (d * c)) = 2 \wedge b = b * (b * 2) = 1$.

Meng and Xin [3] introduced notions of atoms and branches in BCI-algebras. An element a of a BCI-algebra X is called an atom if $z * a = 0$ implies $z = a$ for all $z \in X$. The set of all atoms of X is denoted by $L(X)$. For all a in $L(X)$, $V(a) = \{x \in X | a \leq x\}$ is called a branch of X. Obviously $L(X)$ is the set of all minimal elements of X, and $V(0) = B(X)$. We observe from [4; Theorem 1] that a is an atom of X if and only if $a = x * (x * a)$ for all $x \in X$.

LEMMA 6. ([3]) *Let X be a BCI-algebra. If a and b are atoms of X, then $x \in V(a)$ and $y \in V(b)$ imply $x * y \in V(a * b)$, and $V(a) \cap V(b) = \emptyset$ whenever $a \neq b$.*

Meng and Xin [4] showed that if x and y belong to the same branch of a commutative BCI-algebra X, then $x \wedge y = y \wedge x$. By using Lemma 6 we prove the converse.

THEOREM 7. *Let X be a commutative BCI-algebra. If x and y in X satisfy $x \wedge y = y \wedge x$, then they belong to the same branch of X.*

PROOF. Let $x, y \in X$ be such that $x \wedge y = y \wedge x$. Then, by [4; Corollary 2], there exist $a, b \in L(X)$ such that $x \in V(a)$ and $y \in V(b)$. It follows from Lemma 6 that $x \wedge y = y * (y * x) \in V(b * (b * a)) = V(a)$ and, by similar method, $y \wedge x \in V(b)$, so that $x \wedge y = y \wedge x \in V(a) \cap V(b)$. Using Lemma 6 again, we conclude that $a = b$, whence $V(a) = V(b)$. The proof is complete.

COROLLARY 8. *Let X be a $\wedge$-associative BCI-algebra and let $x, y \in X$. Then $x \wedge y = y \wedge x$ if and only if x and y belong to the same branch of X.*

A BCI-algebra X is said to be locally bounded ([4]) if every branch of X has a greatest element. For each a in $L(X)$, denote by m_a the greatest element of $V(a)$. From [4; Theorems 6, 7, 10 and 11, and Corollary 8], we have the following:

THEOREM 9. *Let X be a $\wedge$-associative BCI-algebra. Then every branch of X is a lower semilattice. If X is locally bounded, then for every a in $L(X)$,*

(i) *$V(a)$ is a distributive lattice.*
(ii) *if we define a mapping $f_a : V(a) \to B(X)$ by $f_a(x) = m_a * x$ for all $x \in V(a)$, then f_a is converse ordered and $f_a(V(a)) = \{f_a(x) | x \in V(a)\}$ is a bounded commutative subalgebra of $B(X)$ with the largest element $m_a * a$.*
(iii) *if $B(X)$ is a chain, then $V(a)$ is also a chain.*

(iv) $V(a) = m_a * B(X)$, *where* $m_a * B(X) = \{m_a * x | x \in B(X)\}$.

THEOREM 10. *An algebra* $(X; *, 0)$ *of type* $(2,0)$ *is a* $\wedge$*-associative BCI-algebra if and only if it satisfies*

(I) $((x * y) * (x * z)) * (z * y) = 0$,

(9) $(x \wedge y) \wedge x = x \wedge y$,

(10) $(x \wedge y) \wedge z = x \wedge (y \wedge z)$,

(1) $x * 0 = x$.

PROOF. Suppose that X satisfies the above conditions. Then X is a commutative BCI-algebra by [4; Theorem 12]. It follows from (10) that X is $\wedge$-associative. The converse is clear.

COROLLARY 11. *The class of* $\wedge$*-associative BCI-algebras is a variety. It is a proper subvariety of the variety of commutative BCI-algebras. The variety of p-semisimple BCI-algebras and the variety of commutative BCK-algebras are proper subvarieties of the variety of* $\wedge$*-associative BCI-algebras.*

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