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INJECTIVE COVER OVER HEREDITARY AND NOETHERIAN RINGS

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ABSTRACT. Using the dual of a categorical definition of an injective envelope, Enochs defined an injective cover. In this paper we will show how injective covers can be used to characterize several well known classes of rings.

1. Introduction

The notion of an injective cover was defined by Enochs in [3]. Although injective cover do not always exist, they do over rings with nice properties (see Theorem 1.2) and have proved to be a useful tool in homological algebra. In this paper we will show how injective covers can be used to characterize several well known classes of rings. Some information about the structure of the injective cover of modules both general and in some special cases can be found in Enochs[3], Ashan,Enochs[1], Cheatham,Enochs,Jenda[2] and Park[4].

DEFINITION 1.1. An injective cover of a left R-module M is a linear map $\phi : E \to M$ with E injective such that:

(1) Any diagram

$$\begin{array}{ccc} & & E' \\ & & \downarrow \\ E & \xrightarrow{\phi} & M \end{array}$$

with E' injective can be completed.

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(2) The diagram

$$\begin{array}{ccc} & & E \\ & & \downarrow{\scriptstyle \phi} \\ E & \xrightarrow{\ \phi\ } & M \end{array}$$

can be completed only by automorphism of E.

Note that $\phi: E \to M$ is called an injective precover if ϕ satisfies (1) only.

THEOREM 1.2. *A ring R is left noetherian if and only if every left R-module has an injective cover.*

PROOF. See Theorem 2.1 [3]

2. Injective covers over hereditary and noetherian rings

PROPOSITION 2.1. *Let R be a left hereditary and left noetherian ring, then every left R-module M has a largest injective submodule $E \subset M$.*

PROOF. For each $i \in I$, let E_i be an injective submodule of M. Let $f : \bigoplus_{i \in I} E_i \to \sum_{i \in I} E_i$ by $f((a_i)) = \sum_{i \in I} a_i$.

Then f is an onto linear map. Since R is a left noetherian, $\bigoplus_{i \in I} E_i$ is also an injective module. Since f is onto, $\sum_{i \in I} E_i \cong \frac{\bigoplus_{i \in I} E_i}{\ker (f)}$. But since R is a left hereditary, $\frac{\bigoplus_{i \in I} E_i}{\ker (f)}$ is an injective module. So $\sum_{i \in I} E_i$ is an injective submodule of M and in fact a largest injective submodule of M if the $\{E_i\}_{i \in I}$ includes all the injective submodules of M.

THEOREM 2.2. *Let R be a left hereditary and left noetherian ring, then for every left R-module M, $i : E \to M$ where E is a largest injective submodule of M and i is a canonical injection is an injective cover of M.*

PROOF. Consider the following diagram

$$\begin{array}{ccc} E' & & \\ & \overset{\phi}{\searrow} & \\ E & \hookrightarrow & M \end{array}$$

where $\phi: E' \to M$ is a linear map for an injective module E'. Since R is hereditary, $\phi(E') \cong \frac{E'}{\ker(\phi)}$ is an injective submodule of M. But E is a largest submodule of M. So we have $\phi(E') \subset E$. So there is a linear map ${}_E|\phi: E' \to E$. Hence we have the following commutative diagram

$$\begin{array}{ccc} E' & & \\ {\scriptstyle {}_E|\phi} \downarrow & \overset{\phi}{\searrow} & \\ E & \hookrightarrow & M \end{array}$$

And the following diagram

$$\begin{array}{ccc} E & & \\ & \overset{i}{\searrow} & \\ E & \hookrightarrow & M \end{array}$$

can be completed only by identity map of E. So $i: E \to M$ is an injective cover of M.

PROPOSITION 2.3. *Let R be a ring such that*
1) every left R-module has an injective cover
2) every injective cover $\phi: E \to M$ is an injection,
then R is a left noetherian and left hereditary.

PROOF. To show R is a left noetherian we argue that for each family of injective modules (E_i), $i \in I$, $\bigoplus_{i\in I} E_i$ is injective. Since $\bigoplus_{i\in I} E_i$ is a module, we have $\phi: E \to \bigoplus_{i\in I} E_i$ an 1-1 injective cover. So $E \cong \phi(E) \subset \bigoplus_{i\in I} E_i$, so we can assume E is a submodule of $\bigoplus_{i\in I} E_i$ and that ϕ is the canonical injection. So by hypothesis we have the following commutative diagram

$$\begin{array}{ccc} E_i & & \\ {}_E|\sigma \downarrow & \overset{\sigma}{\searrow} & \\ E & \hookrightarrow & \bigoplus_{i\in I} E_i \end{array}$$

Clearly $E \subset \bigoplus_{i\in I} E_i$. So let $a \in \bigoplus_{i\in I} E_i$.

Then w.l.o.g. $a = (a_1, a_2, ..., a_n,\ 0, 0\cdots)$. So consider the following commutative diagram

$$\begin{array}{ccc} E_1 \oplus E_2 \oplus \cdots E_n & & \\ {}_E|\sigma \downarrow & \overset{\sigma}{\searrow} & \\ E & \hookrightarrow & \bigoplus_{i\in I} E_i \end{array}$$

Then $a \in \sigma(E_1 \oplus E_2 \oplus \cdots \oplus E_n) \subset E$. So $a \in E$. So $E = \bigoplus_{i\in I} E_i$. Hence $\bigoplus_{i\in I} E_i$ is an injective module. So R is a left noetherian. Now we show that R is left hereditary. Let E be an injective module and $S \subset E$ be a submodule. Then E/S is a module. So we have an one to one injective cover $\phi :\ F \to E/S$. So $F \cong \phi(F) \subset E/S$, so we can assume F is a submodule of E/S and that ϕ is the inclusion map. So we have the following commutative diagram

$$\begin{array}{ccc} E & & \\ {}_F|\sigma \downarrow & \overset{\sigma}{\searrow} & \\ F & \hookrightarrow & E/S \end{array}$$

Clearly $F \subset E/S$. So let $a + S \in E/S$. Then $a + S \in \sigma(E) \subset F$. Thus $F = E/S$. Therefore E/S is an injective module. Hence R is a left hereditary.

DEFINITION 2.4. A ring R is left Quasi-Frobenius if it is left noetherian and R is an injective left R-module.

PROPOSITION 2.5. *A ring R is left Quasi-Frobenius if and only if*
1) *every left R-module has an injective cover*
2) *every injective cover $\phi: E \to M$ is a surjection.*

PROOF. ($\Rightarrow$) Since R is left noetherian, every left R-module has an injective cover. Let M be a left R-module and $\phi: E \to M$ be an injective cover of M. Let $x \in M$ then since $_RR$ is free module with base $\{1\}$ we have a linear map $\sigma: R \to M$ such that $\sigma(1) = x$. So we have the following commutative diagram

$$\begin{array}{ccc} R & & \\ \tau\downarrow & \overset{\sigma}{\searrow} & \\ E & \overset{\phi}{\to} & M \end{array}$$

such that $\phi(\tau(1)) = \sigma(1) = x$. So we have $x'' = \tau(1) \in E$ such that $\phi(x'') = x$. So ϕ is onto.

($\Leftarrow$) Since every left R-module has an injective cover, R is left noetherian. Since R is free module R is projective. And by assumption we have an onto linear map $\phi: E \to R$. So we have the following commutative diagram

$$\begin{array}{ccc} & R & \\ \overset{\tau}{\nearrow} & \downarrow id & \\ E \overset{\phi}{\to} & R & \to 0 \end{array}$$

Therefore τ is one to one. Thus $E = R \oplus \ker(\phi)$. But E is injective so is R. Hence R is left Quasi-Frobenius.

The follwoing Theorem is well known, we just state this Theorem without proof.

THEOREM 2.6. *The following statements are equivalent.*
1) *A ring R is left noetherian, left hereditary and R is an injective left$R$-module.*
2) *All modules are injective.*
3) *All modules are projective.*

THEOREM 2.7. *Suppose $\phi : E \to M$ is an injective cover and that $\phi' : E' \to M$ is an injective precover. Then there is a linear map $f : E \to E'$ such that $\phi' \circ f = \phi$ with f an injection and that $f(E)$ is a direct summand of E'.*

PROOF. Since $\phi' : E' \to M$ is an injective precover the following diagram

$$\begin{array}{ccc} E & & \\ & \overset{\phi}{\searrow} & \\ E' & \overset{\phi'}{\to} & M \end{array}$$

can be completed by a linear map $f : E \to E'$ such that $\phi' \circ f = \phi$. Consider the following commutative diagram

$$\begin{array}{ccc} E & & \\ f \downarrow & \overset{\phi}{\searrow} & \\ E' & \overset{\phi'}{\to} & M \\ g \downarrow & \underset{\phi}{\nearrow} & \\ E & & \end{array}$$

Since ϕ is an injective cover $g \circ f \in Aut_E$. So f is an injective. Now consider the following commutative diagram

$$\begin{array}{ccc} & & E \\ & \overset{f}{\nearrow} & \downarrow_{a \in Aut_E} \\ E' & \underset{g}{\to} & E \end{array}$$

Let $x \in f(E) \cap \ker(g)$. Then since f is an injective there is $y \in E$ such that $f(y) = x$. But $g(x) = 0$ so $a(y) = g(f(y)) = g(x) = 0$ so $y = 0$ since $a \in Aut_E$. So $x = f(y) = f(0) = 0$. To show $E' = f(E) + \ker(g)$, let $x \in E'$, then $g(x) = y \in E$. Since $a = Aut_E$ there is $z \in E$ such that $a(z) = y$. And we have $f(z) = w \in f(E)$ and that $g(w) = y$. Now $x = w + (x - w)$, but $w \in f(E)$ and $g(x - w) = g(x) - g(w) = y - y = 0$. So $x - w \in \ker(g)$. So $E' = f(E) + \ker(g)$. So $E' = f(E) \oplus \ker(g)$. So $E' \cong E \oplus \ker(g)$.

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