Comm. Korean Math. Soc. **10** (1995), No. 1, pp. 251–259

THRESHOLD RESULTS FOR THE MCKEAN EQUATION

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1. Introduction

In 1952, British physiologists Hodgkin and Huxley [4] derived a model that describes the conduction of the nervous impulse in the optical nerve of a squid. The mathematical analysis of the Hodgkin-Huxley equations is technically very difficult, because of the complicated nonlinear functions in the equations. In the early 1960's FitzHugh and Nagumo [2], [9] derived a simpler formulation which retains most of the qualitative features of the original system, and yet is more amenable to analytical manipulations. The equation is

$$\begin{aligned} v_t &= v_{xx} + f(v) - w \\ w_t &= \epsilon(v - \gamma w), \end{aligned} \tag{1.1}$$

where $f(v) = v(1-v)(v-a)$ $(0 < a < 1)$, $\epsilon \geq 0$, and $\gamma \geq 0$. McKean [6], [7] suggested a further simplification in which $f(v) = v(1-v)(v-a)$ is replaced by $f(v) = -v + H(v-a)$, where H is the Heaviside step function.

In this paper we consider the initial value problem for the equation

$$v_t = v_{xx} + f(v), \tag{1.2}$$

the initial datum being $v(x,0) = \varphi(x)$. We assume that $f(v) = -v + H(v-a)$, where H is the Heaviside step function and $a \in (0, 1/2)$. Note that (1.2) is obtained by setting $\epsilon = 0$ and $w \equiv 0$ at (1.1).

Our primary interest is to study the asymptotic behavior of solutions of (1.2). One expects (1.2) to exhibit a threshold phenomenon. That is, if the initial datum is sufficiently small, then one expects the solutions

Received October 4, 1994. Revised November 18, 1994.
Key words: McKean equation, threshold phenomena, initial value problem.

of (1.2) to decay exponentially fast to zero as t goes to infinity. In this case, we say $\varphi(x)$ is subthreshold. This corresponds to the biological fact that a minimum amount of stimulus is needed to trigger a nerve impulse. One expects, however, that if $\varphi(x)$ is sufficiently large, or superthreshold, then some sort of signal will propagate. Threshold results for the equation (1.2) with smooth "cubic-like" function f have been given by Aronson and Weinberger [1]. Terman [11] showed if $\varphi(x) > a$ on a sufficiently large interval, then $\varphi(x)$ is superthreshold in the equation (1.2).

It was thought [3] that if $\varphi(x)$ has a small compact support, that is, $\varphi(x) \equiv 0$ outside some interval $[-d, d]$ for small d, then the integral of $\varphi(x)$ is a crucial factor in the threshold phenomenon. In this paper we give a rigorous mathematical proof of this fact in the equation (1.2).

Throughout this paper we assume that the initial datum $\varphi(x)$ satisfies the following conditions:

(a) $\varphi(x) \in C^1(R)$, (b) $\varphi(x) \in [0, \infty)$ in R,
(c) $\varphi(x) = \varphi(-x)$ in R, (d) $\varphi'(x) \leq 0$ in R^+,
(e) $\lim_{x \to +\infty} \varphi(x) = 0$.

By a classical solution of equation (1.2) we mean the following:

DEFINITION. Let $S_T = R \times (0, T)$ and $G_T = \{(x,t) \in S_T, v(x,t) \neq a\}$. Then $v(x,t)$ is said to be a classical solution of the Cauchy problem (1.2) if

(a) The functions v and v_x are bounded, continuous in S_T.

(b) The functions v_{xx} and v_t are continuous in G_T, and satisfy the equation

$$v_t = v_{xx} + f(v).$$

(c) $\lim_{t \to 0} v(x,t) = \varphi(x)$ for each $x \in R$.

If $\varphi(0) < a$, then $v(x,t) < a$ in $R \times R^+$ by the maximum principle [10, pp. 159-72]. Hence v satisfies the linear partial differential equation $v_t = v_{xx} - v$. Therefore $v(x,t) \leq e^{-t}\varphi(0)$, so $v(x,t)$ decays exponentially fast to 0. This is a simple case for the initial datum to be subthreshold. Throughout this paper we assume that $\varphi(0) \geq a$. We consider the curve $s(t)$ defined by

$$s(t) = \sup\{x : v(x,t) = a\}. \tag{1.3}$$

We say that the initial datum is superthreshold if $\lim_{t\to\infty} s(t) = +\infty$, and subthreshold if $s(t)$ is bounded above by a constant x_0 for all $t \geq 0$.

Since $\varphi'(x) \leq 0$ in R^+, we expect that $v_x(x,t) < 0$ in $R^+ \times R^+$. Therefore $s(t)$ is a well defined, continuous function for some time, say $t \in [0,T]$. It follows that $v > a$ for $|x| < s(t)$, and $v < a$ for $|x| > s(t)$. Let χ_Ω be the indicator function of the subset

$$\Omega = \{(x,t) : v(x,t) > a, 0 \leq t \leq T\}. \tag{1.4}$$

Then $v(x,t)$ satisfies the inhomogeneous equation

$$v_t = v_{xx} - v + \chi_\Omega \text{ for } |x| \neq s(t). \tag{1.5}$$

By Duhamel's principle, the solution can be expressed as

$$v(x,t) = \int_{-\infty}^{\infty} K(x-\xi,t)\varphi(\xi)d\xi + \int_0^t d\tau \int_{-s(\tau)}^{s(\tau)} K(x-\xi,t-\tau)d\xi, \tag{1.6}$$

where $K(x,t) = (e^{-t}/2\sqrt{\pi t})e^{-x^2/4t}$ is the fundamental solution of the differential equation $\psi_t = \psi_{xx} - \psi$. Setting $x = s(t)$ in (1.6), we have the integral equation

$$a = \int_{-\infty}^{\infty} K(s(t)-\xi,t)\varphi(\xi)d\xi + \int_0^t d\tau \int_{-s(\tau)}^{s(\tau)} K(s(t)-\xi,t-\tau)d\xi. \tag{1.7}$$

The following theorem shows that the solution of (1.2) is completely determined by the curve $s(t)$.

THEOREM 1.1. *Suppose that $s(t)$ is a continuously differentiable function which satisfies the integral equation (1.7) in $[0,T]$, then the function $v(x,t)$ given by (1.6) is a classical solution of the equation (1.2) in $R \times [0,T]$.*

Proof. See [11].

2. Lower and upper solutions

Let $\psi(x,t)$ be the solution of the initial value problem

$$\psi_t = \psi_{xx} - \psi, \ (x,t) \in R \times R^+ \tag{2.1}$$
$$\psi(x,0) = \varphi(x), \ x \in R.$$

Then $\psi(x,t) = \int_{-\infty}^{\infty} K(x-\xi,t)\varphi(\xi)d\xi$. Assume $\alpha(t)$ is a nonnegative, uniformly Lipschitz continuous function defined for $t \in [0,T]$. We define the functions $\Phi(\alpha)(t)$ and $\Psi(\alpha)(t)$ on the interval $[0, T]$ by

$$\Phi(\alpha)(t) = \int_0^t d\tau \int_{-\alpha(\tau)}^{\alpha(\tau)} K(\alpha(t)-\xi, t-\tau)d\xi, \tag{2.2}$$

$$\Psi(\alpha)(t) = \psi(\alpha(t),t). \tag{2.3}$$

Note that $\lim_{t\to 0}\Psi(\alpha)(t) = \varphi(\alpha(0))$.

If $\Phi(\alpha)(t) + \Psi(\alpha)(t) \geq a$ on $[0,T]$, then we call $\alpha(t)$ a *lower solution* on $[0,T]$. If $\Phi(\alpha)(t) + \Psi(\alpha)(t) \leq a$ on $[0,T]$, then $\alpha(t)$ is called an *upper solution* on $[0,T]$.

REMARKS. **1.** $\alpha(t)$ is a solution of the integral equation (1.7) if and only if $\alpha(t)$ is a lower and upper solution.

2. If $\alpha(t)$ is a lower solution, then $\varphi(\alpha(0)) \geq a$, hence $\alpha(0) \leq s(0)$. In the same way, if $\alpha(t)$ is an upper solution, then $\alpha(0) \geq s(0)$.

3. If $\alpha(t)$ and $\beta(t)$ are respectively lower and upper solution on $[0,T]$ and $\alpha(0) < \beta(0)$, then $\alpha(t) < \beta(t)$ on $[0,T]$.

In this section we show some properties of the function Φ.

LEMMA 2.1. *Suppose* $\alpha(t) = x_0$ *be a vertical line* ($x_0 > 0$). *Set* $\Phi(x_0) = \lim_{t\to\infty}\Phi(\alpha)(t)$, *then the function* $\Phi(x_0)$, *defined for* $0 < x_0 < \infty$, *satisfies the following:*

(a) $\lim_{x_0\to 0}\Phi(x_0) = 0$,

(b) $\lim_{x_0\to\infty}\Phi(x_0) = 1/2$,

(c) $\Phi'(x_0) > 0$ *for* $0 < x_0 < \infty$.

Proof. By the definition of Φ, we have

$$\begin{aligned}\Phi(\alpha)(t) &= \int_0^t d\tau \int_{-\alpha(\tau)}^{\alpha(\tau)} K(\alpha(t)-\xi, t-\tau)d\xi \\ &= \int_0^t d\tau \int_{-x_0}^{x_0} K(x_0-\xi, t-\tau)d\xi.\end{aligned}$$

Using the change of variables $\eta = t - \tau$, we have

$$\Phi(\alpha)(t) = \int_0^t d\eta \int_{-x_0}^{x_0} K(x_0 - \xi, \eta) d\xi.$$

Therefore

$$\Phi(x_0) = \int_0^\infty d\eta \int_{-x_0}^{x_0} K(x_0 - \xi, \eta) d\xi$$

$$= \int_0^\infty d\eta \int_{-2x_0}^{0} K(-\xi, \eta) d\xi. \tag{2.4}$$

Now the proofs of (a) and (c) easily follow from (2.4). The proof of (b) follows from the computations

$$\int_0^\infty d\eta \int_{-\infty}^{0} K(-\xi, \eta) d\xi = \int_0^\infty 1/2 e^{-\eta} d\eta = 1/2.$$

This completes the proof the lemma.

LEMMA 2.2. *Suppose $\alpha(t) = ct$ is a linear function ($c > 0$). Set $\Phi(c) = \lim_{t\to\infty} \Phi(\alpha)(t)$, then the function $\Phi(c)$, defined for $0 < c < \infty$, satisfies the following:*

(a) $\lim_{c\to 0} \Phi(c) = 1/2$,

(b) $\lim_{c\to\infty} \Phi(c) = 0$,

(c) $\Phi'(c) < 0$ *for* $0 < c < \infty$.

Proof. We have

$$\Phi(\alpha)(t) = \int_0^t d\tau \int_{-\alpha(\tau)}^{\alpha(\tau)} K(\alpha(t) - \xi, t - \tau) d\xi$$

$$= \int_0^t d\tau \int_{-c\tau}^{c\tau} K(ct - \xi, t - \tau) d\xi.$$

Using the change of variables $\eta = t - \tau$, $\zeta = \xi - ct$, we have

$$\Phi(\alpha)(t) = \int_0^t d\eta \int_{c\eta - 2ct}^{-c\eta} K(-\zeta, \eta) d\zeta.$$

Therefore

$$\Phi(c) = \int_0^\infty d\eta \int_{-\infty}^{-c\eta} K(-\zeta, \eta) d\zeta. \tag{2.5}$$

The proof of the lemma easily follows from (2.5).

3. Existence and uniqueness

In this section we state some results of existence and uniqueness of solution $s(t)$ of (1.7).

THEOREM 3.1. *Assume that there exist linear functions* $\alpha(t)$ *and* $\beta(t)$, *which are respectively lower and upper solutions on* $[0,T]$ *for some positive time* T, *and* $e^{-\alpha(T)/T} \leq 1/4$, *then there exists a solution* $s(t)$ *of the integral equation* (1.7) *on* $[0,T]$.

Proof. See [11], and [8] for other existence results.

We assume in this paper that the solution $s(t)$ exists in all of R^+. The following theorem demonstrates that the solution $s(t)$ of (1.7) is unique among uniformly Lipschitz functions.

THEOREM 3.2. *Suppose that* $\alpha(t)$ *and* $\beta(t)$ *are respectively lower and upper solutions on* $[0,T]$, *then* $\alpha(t) \leq \beta(t)$ *on* $[0,T]$.

Proof. See [11].

4. The main theorem

Let a be a fixed constant in $(0,1/2)$. We assume the initial datum $\varphi(x)$ has a compact support. We denote the support of $\varphi(x)$ by $S(\varphi)$, and the integral $\int_{-\infty}^{\infty} \varphi(x)dx$ by $A(\varphi)$. First, we prove the superthreshold result.

THEOREM 4.1. *For any* $d^* > 0$, *there exist* $M^*(d^*)$ *such that if* $\varphi(x)$ *is a function satisfying* $S(\varphi) \subset [-d^*, d^*]$ *and* $A(\varphi) > M^*$, *then* $\varphi(x)$ *is superthreshold.*

Proof. We can choose c such that $\Phi(c) > a$ by Lemma 2.2. Let $\alpha(t)$ be the curve defined by

$$\alpha(t) = \begin{cases} 0 & \text{for} \quad 0 \leq t < t_1 \\ \gamma(t) & \text{for} \quad t_1 \leq t \leq t_2 \\ ct & \text{for} \quad t > t_2. \end{cases}$$

Here t_1 and t_2 are some numbers such that $0 < t_1 < t_2$, and $\gamma(t)$ is a curve defined on $[t_1, t_2]$ which connects the two curves smoothly. We can easily show

$$\lim_{t\to\infty} \Phi(\alpha)(t) = \Phi(c).$$

Therefore we can find T such that $\Phi(\alpha)(t) \geq a$ for $t \geq T$.

First, we estimate $\Psi(\alpha)(t)$ for $t \in (0, t_1]$.

$$\begin{aligned}\Psi(\alpha)(t) &= \int_{-\infty}^{\infty} K(\alpha(t) - \xi, t)\varphi(\xi)d\xi \\ &= \int_{-d^*}^{d^*} K(-\xi, t)\varphi(\xi)d\xi \\ &\geq \int_{-d^*}^{d^*} K(-d^*, t)\varphi(\xi)d\xi \\ &= K(-d^*, t)A(\varphi).\end{aligned}$$

For a given time $t > 0$, the function $\psi(x,t)$ in (2.1) takes its maximum at $x = 0$. Therefore, from the maximum principle, $\Psi(\alpha)(t)$ is a decreasing function of t on $[0, t_1]$. Hence, on the interval $[0, t_1]$, we have

$$\Psi(\alpha)(t) \geq K(-d^*, t_1)A(\varphi).$$

Next, we estimate $\Psi(\alpha)(t)$ in the interval $[t_1, T]$.

$$\begin{aligned}\Psi(\alpha)(t) &= \int_{-\infty}^{\infty} K(\alpha(t) - \xi, t)\varphi(\xi)d\xi \\ &= \int_{-d^*}^{d^*} K(\alpha(t) - \xi, t)\varphi(\xi)d\xi \\ &\geq \int_{-d^*}^{d^*} K(\alpha(t) + d^*, t)\varphi(\xi)d\xi \\ &= K(\alpha(t) + d^*, t)A(\varphi).\end{aligned}$$

Put $m = inf_{t_1 \leq t \leq T}\{K(\alpha(t) + d^*, t)\} > 0$. Now we choose M^* any number bigger than $max\{a/K(-d^*, t_1), a/m\}$. Suppose $\varphi(x)$ be a function such that $S(\varphi) \subset [-d^*, d^*]$ and $A(\varphi) > M^*$. Then $\Psi(\alpha)(t) \geq a$ on $[0, T]$. Since $\Phi(\alpha)(t) \geq a$ for $t \geq T$, we have

$$\Phi(\alpha)(t) + \Psi(\alpha)(t) \geq a, \text{ for } t > 0.$$

Hence $\alpha(t)$ is a lower solution. We have $s(t) \geq \alpha(t)$ for $t \geq 0$ by Theorem 3.2. Therefore $\lim_{t\to\infty} s(t) = +\infty$. Thus $\varphi(x)$ is superthreshold. This completes the proof of the theorem.

Next, we prove the subthreshold result.

THEOREM 4.2. *There exist positive constants d^* and m^* such that if $\varphi(x)$ is a function satisfying $S(\varphi) \subset [-d^*, d^*]$ and $A(\varphi) < m^*$, then $\varphi(x)$ is subthreshold.*

Proof. We can choose $x_0 > 0$ such that $0 < \Phi(x_0) < a$ by Lemma 2.1. Choose a positive number $d^* < x_0$. Set $\alpha(t) = x_0$ be a vertical line. Then, for $t \geq 0$

$$\begin{aligned}\Phi(\alpha)(t) &= \int_0^t d\tau \int_{-x_0}^{x_0} K(x_0 - \xi, t - \tau) d\xi \\ &\leq \Phi(x_0).\end{aligned}$$

Suppose $\varphi(x)$ be a function such that $S(\varphi) \subset [-d^*, d^*]$. Then

$$\begin{aligned}\Psi(\alpha)(t) &= \int_{-\infty}^{\infty} K(x_0 - \xi, t)\varphi(\xi) d\xi \\ &= \int_{-d^*}^{d^*} K(x_0 - \xi, t)\varphi(\xi) d\xi \\ &\leq K(x_0 - d^*, t) A(\varphi).\end{aligned}$$

It is clear that

$$\lim_{t\to\infty} K(x_0 - d^*, t) = 0, \ \lim_{t\to 0} K(x_0 - d^*, t) = 0.$$

Put $L = \sup_{0<t<\infty} K(x_0 - d^*, t) < \infty$. We choose m^* a number less than $(a - \Phi(x_0))/L$. Now, if $A(\varphi) < m^*$, then

$$\Phi(\alpha)(t) + \Psi(\alpha)(t) \leq \Phi(x_0) + a - \Phi(x_0) = a,$$

for all $t \geq 0$. Hence $\alpha(t) = x_0$ is an upper solution. Therefore $\varphi(x)$ is subthreshold. Now the proof is complete.

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