Comm. Korean Math. Soc. **10** (1995), No. 1, pp. 235–249

ON THE M–SOLUTION OF THE FIRST KIND EQUATIONS

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1. Introduction

Let K be a bounded linear operator from Hilbert space H_1 into Hilbert space H_2. When numerically solving the first kind equation $Kf = g$, one usually picks n orthonormal functions $\phi_1, \phi_2, \dots, \phi_n$ in H_1 which depend on the numerical method and on the problem, see Varah [12] for more details. Then one finds the unique minimum norm element $f_M \in M$ that satisfies $\|Kf_M - g\| = \inf\{ \|Kf - g\| : f \in M\}$, where M is the linear span of $\phi_1, \phi_2, \dots, \phi_n$. Such a solution $f_M \in M$ is called the M-solution of $Kf = g$. Some methods for finding the M-solution of $Kf = g$ were proposed by Banks [2] and Marti [9, 10]. See [5, 6, 8] for convergence results comparing the M-solution of $Kf = g$ with f_0, the least squares solution of minimum norm (LSSMN) of $Kf = g$.

Throughout this paper, it is assumed that H_1 and H_2 are Hilbert spaces and M is a finite dimensional subspace of H_1. Let $B(H_1, H_2)$ denote the space of all bounded linear operators from H_1 to H_2, and for $K \in B(H_1, H_2)$ let K_N denote the restriction of K to a subspace N of H_1, i.e., $K_N = K|_N : N \to H_2$ is an operator such that $K_N x = Kx$ for all $x \in N$. For $K \in B(H_1, H_2)$, K^* denotes the adjoint operator in $B(H_2, H_1)$ and $K^\dagger : R(K) + R(K)^\perp \to H_1$ denotes the Moore-Penrose generalized inverse of K [4], where $R(K)$ refers to the range space of K and $R(K)^\perp$ refers to the orthogonal complement of $R(K)$. Let $(\cdot, \cdot)$ denote an inner product and $\|\cdot\|$ denote a corresponding norm, and let $\langle \phi_1, \phi_2, \dots, \phi_n \rangle$ denote the closed linear span of $\phi_1, \phi_2, \dots, \phi_n$.

Received June 20, 1994. Revised November 7, 1994.

AMS Classification: 65R20, 65J10, 45B45, 45L10.

Key word: M-solution, first kind equation well-posed.

This paper was supported (in part) by Chungbuk National University Research Foundation, 1993.

It is well-known that the problem of solving the first kind equation $Kf = g$ is ill-posed in that arbitrarily small perturbations in the data g or K may cause an arbitrarily large perturbation in the solution f [1, 11]. A typical example of such an ill-posed problem is the Fredholm integral equation of the first kind. The purpose of this paper is to develop a new method producing the M-solution to $Kf = g$ which is well-posed under perturbations in both K and g.

This paper is organized as follows. First, it is shown that the M-solution of $Kf = g$ is *well-posed* under perturbation in g, but is *not well-posed* under perturbations in both K and g. Then, a generalized Gram-Schmidt (GGS) method for finding the M-solution of $Kf = g$ is proposed, and it is shown that a *modified version* of the GGS method, called the GGS(δ), provides the M-solution to $Kf = g$ which is well-posed under perturbations in both K and g. Lastly, numerical implementation of the GGS(δ) method is described in detail, and then some numerical results obtained by applying this method to the first kind Fredholm integral equations are reported.

2. Numerical method and its well-posedness

We begin this section by giving a precise definition for the well-posedness of the M-solution to $Kf = g$.

DEFINITION 2.1. Let $K \in B(H_1, H_2)$. The M-solution of the first kind equation $Kf = g$ is *well-posed under perturbations in both K and g* if for each $\epsilon > 0$, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that for all $\hat{g} \in H_2$ and $\hat{K} \in B(H_1, H_2)$ with $\|g - \hat{g}\| < \delta_1$ and $\|K - \hat{K}\| < \delta_2$, $\|f_M - \hat{f}_M\| < \epsilon$, where f_M and $\hat{f}_M$ are the M-solutions of $Kf = g$ and $\hat{K}f = \hat{g}$, respectively. That is, if the M-solution of $Kf = g$ depends continuously on the data K and g, then the M-solution is said to be well-posed under perturbations in both K and g.

In the above definition, if K is fixed and only g is allowed to vary, then the M-solution of $Kf = g$ is said to be *well-posed under perturbation in g*. Similarly, the well-posedness of the M-solution to $Kf = g$ under perturbation in K can be defined.

THEOREM 2.2. *The M-solution of $Kf = g$ is well-posed under perturbation in g, but it is not well-posed under perturbation in K, where $K \in B(H_1, H_2)$.*

Proof. Since M is finite dimensional, K_M has closed range. Thus $K_M^\dagger$ is a bounded linear operator. For an element $\hat{g}$ in H_2, let f_M and $\hat{f}_M$ denote the M-solutions of $Kf = g$ and $Kf = \hat{g}$, respectively. Then

$$\|\hat{f}_M - f_M\| = \|K_M^\dagger \hat{g} - K_M^\dagger g\| \le \|K_M^\dagger\| \, \|\hat{g} - g\|.$$

Therefore, the M-solution is well-posed under perturbation in g.

To see that the M-solution is not well-posed under perturbation in K, let $H_1 = H_2$, ϕ_1 and ϕ_2 be orthonormal vectors in H_1, $M = \langle \phi_1, \phi_2 \rangle$, and let P_i be the orthogonal projection of H_1 onto $\langle \phi_i \rangle$ for $i = 1, 2$. Then define $K = P_1$ and $K^{(n)} = P_1 + \frac{1}{n} P_2$, where n is a natural number. For $g = \alpha_1 \phi_1 + \alpha_2 \phi_2$ with $\alpha_2 \neq 0$, let f_M and $f_M^{(n)}$ denote the M-solutions of $Kf = g$ and $K^{(n)} f = g$, respectively. Then, we have

$$\begin{aligned} \|f_M^{(n)} - f_M\| &= \|(K_M^{(n)})^\dagger g - K_M^\dagger g\| \\ &= \|(\alpha_1 \phi_1 + n\alpha_2 \phi_2) - \alpha_1 \phi_1\| \\ &= n|\alpha_2|. \end{aligned}$$

Since $\|K - K^{(n)}\| = \frac{1}{n}$, $K^{(n)}$ converges to K as $n \to \infty$. However, from the above equality, $f_M^{(n)}$ does not converge to f_M as $n \to \infty$. This completes the proof.

The above theorem implies that the M-solution of the first kind equation $Kf = g$ is not well-posed under perturbations in both K and g. We now introduce a generalized Gram-Schmidt (GGS) method for calculating the M-solution of $Kf = g$ which is based on the Gram-Schmidt orthogonalization procedure, where $K \in B(H_1, H_2)$. From now on, I denotes an index set $\{1, 2, \ldots, n\}$. Let $\{\phi_i\}_{i \in I}$ be an orthonormal basis for M. The GGS method on K relative to M consists of two steps. The **first step** of the GGS method is as follows:

$$\text{Compute } y_1 = K\phi_1$$

$$\text{Let } \psi_1 = \begin{cases} \frac{y_1}{\|y_1\|} & \text{if } y_1 \neq 0 \\ 0 & \text{if } y_1 = 0 \end{cases}$$

For $i = 2, \ldots, n$

$$y_i = K\phi_i - \sum_{j=1}^{i-1}(K\phi_i, \psi_j)\psi_j$$

$$\text{Let } \psi_i = \begin{cases} \frac{y_i}{\|y_i\|} & \text{if } y_i \neq 0 \\ 0 & \text{if } y_i = 0 \end{cases}$$

From this step, it can be easily seen that $\|\psi_i\| = \begin{cases} 1 & \text{if } y_i \neq 0 \\ 0 & \text{if } y_i = 0 \end{cases}$, $(\psi_i, \psi_j) = 0$ for $i \neq j$, and $\langle \psi_1, \ldots, \psi_i \rangle = \langle K\phi_1, \ldots, K\phi_i \rangle$ for each $i \in I$.

To describe the second step of the GGS, we need the following theorem which underlies this step. Define the $n \times n$ matrix $A = (a_{ij})$, where $a_{ij} = (K\phi_j, \psi_i)$, and the vector $b = (b_i) \in R^n$, where $b_i = (g, \psi_i)$. Note that $a_{ij} = 0$ for $i > j$ and $a_{ii} = (K\phi_i, \psi_i) = (y_i, \psi_i) = \|y_i\|$. Recall that the elements of M can be written as $\sum_{j=1}^n \beta_j \phi_j$.

THEOREM 2.3. *$\sum_{j=1}^n \beta_j\phi_j$ is the M-solution of $Kf = g$ if and only if the vector $z = (\beta_j) \in R^n$ is the minimum 2-norm solution of $Ax = b$, i.e., $z = A^\dagger b$, where $A^\dagger$ is the Moore-Penrose generalized inverse of the matrix A* [7].

Proof. Let P denote the orthogonal projection of H_2 onto $R(K_M)$. Suppose that $\tilde{f} = \sum_{j=1}^n \beta_j\phi_j$ is a solution to $Kf = Pg$. Since $(g - Pg) \in \langle K\phi_1, \ldots, K\phi_n\rangle^\perp$, by the construction of $\{\psi_i\}$ $(g - Pg) \in \langle \psi_1, \ldots, \psi_n\rangle^\perp$.
Thus, for each $i \in I$

$$0 = (g - Pg, \psi_i) = (g - K\tilde{f}, \psi_i) = (g, \psi_i) - (K\tilde{f}, \psi_i).$$

It follows that for each $i \in I$

$$b_i = (g, \psi_i) = \sum_{j=1}^n \beta_j(K\phi_j, \psi_i) = \sum_{j=1}^n a_{ij}\beta_j.$$

Hence the vector $z = (\beta_j)$ is a solution of $Ax = b$.

Suppose that $z = (\beta_j)$ is a solution to $Ax = b$. Let $\tilde{f} = \sum_{j=1}^n \beta_j\phi_j$. Since $Az = b$, for each $i \in I$ $b_i = \sum_{j=1}^n a_{ij}\beta_j$ and so $(g, \psi_i) = (K\tilde{f}, \psi_i)$.

Thus, $(g-K\tilde{f},\psi_i)=0$ for each $i\in I$. This implies that $(g-K\tilde{f},K\phi_i)=0$ for each $i\in I$ and hence $(g-K\tilde{f})\in R(K_M)^{\perp}$. Therefore, $\tilde{f}=\sum_{j=1}^{n}\beta_j\phi_j$ is a solution to $Kf=Pg$.

Note that for $\tilde{f}=\sum_{j=1}^{n}\beta_j\phi_j$ and $z=(\beta_j)$, $\|\tilde{f}\|^2=\sum_{j=1}^{n}|\beta_j|^2=\|z\|_2^2$ Since the M-solution of $Kf=g$ is the minimum norm element in M satisfying $Kf=Pg$, from the above arguments the theorem holds.

From this theorem, we can see that the **second step** of the GGS method is to find the minimum norm solution $z=(\beta_j)$ of $Ax=b$ and then form the M-solution $\sum_{j=1}^{n}\beta_j\phi_j$. Notice that the GGS method does not require the linear independence of $K\phi_1,\dots,K\phi_n$, whereas the Banks method [2] does require the linear independence of $K\phi_1,\dots,K\phi_n$.

When one implements the GGS method on a computer with finite precision arithmetic, it is very unlikely to have an exact zero for y_i. This fact is a main motivation for developing a *modified version* of the GGS method which is called the GGS(δ) method from now on, where $\delta>0$ is a fixed small constant. The choice of a suitable constant δ is discussed later. The GGS(δ) method on K relative to M also consists of two steps. The **first step** of the GGS(δ) method is as follows:

Compute $y_1=K\phi_1$

$$\text{Let } \psi_1=\begin{cases}\frac{y_1}{\|y_1\|} & \text{if } \|y_1\|\ \geq\delta\\ 0 & \text{if } \|y_1\|\ <\delta\end{cases}$$

For $i=2,\dots,n$

$$y_i=K\phi_i-\sum_{j=1}^{i-1}(K\phi_i,\psi_j)\psi_j$$

$$\text{Let } \psi_i=\begin{cases}\frac{y_i}{\|y_i\|} & \text{if } \|y_i\|\ \geq\delta\\ 0 & \text{if } \|y_i\|\ <\delta\end{cases}$$

From the above step, it is easy to see that $\|\psi_i\|\ =\begin{cases}1 & \text{if } \|y_i\|\ \geq\delta\\ 0 & \text{if } \|y_i\|\ <\delta\end{cases}$, $(\psi_i,\psi_j)=0$ for $i\neq j$, and $\langle\psi_1,\dots,\psi_i\rangle\subset\langle K\phi_1,\dots,K\phi_i\rangle$ for each $i\in I$ which is not the same as the corresponding one of the GGS. The **second step** of the GGS(δ) is to compute $A^{\dagger}b$ and then form the *approximate*

M-solution $\sum_{j=1}^{n}(A^{\dagger}b)_j\phi_j$ to $Kf = g$, where A and b are defined the same as in the GGS method. From now on, μ refers to the positive real number $\min\{\ \|y_i\|\ :\ \|y_i\|\ > 0,\ i \in I\}$.

If δ is chosen so that $0 < \delta < \mu$, then $\|y_i\|\ < \delta$ is equivalent to $y_i = 0$, and hence the GGS(δ) method is mathematically the same as the GGS method. For this δ, it is clear that $\langle\psi_1, \dots, \psi_i\rangle = \langle K\phi_1, \dots, K\phi_i\rangle$ for each $i \in I$. Therefore, if $0 < \delta < \mu$, then by Theorem 2.3 the approximate M-solution $\sum_{j=1}^{n}(A^{\dagger}b)_j\phi_j$ obtained from the GGS(δ) becomes the (exact) M-solution to $Kf = g$ (it is of course assumed that all operations are carried out using infinite precision arithmetic). When using finite precision arithmetic, choosing a constant δ a priori so that $\sum_{j=1}^{n}(A^{\dagger}b)_j\phi_j$ is the M-solution to $Kf = g$ is not an easy problem, but numerical experiments show that such a δ may be chosen as a real number which is slightly greater than the unit roundoff for the specific computer to be used (see Section 3).

THEOREM 2.4. *Let $K \in B(H_1, H_2)$. If $\delta > 0$ is a fixed constant chosen sufficiently small, then the GGS(δ) method provides the M-solution of $Kf = g$ which is well-posed under perturbations in both K and g. Actually, the assumption of this theorem means that δ may be chosen to be any real number such that $0 < \delta < \mu$.*

For the proof of this theorem, we will use the following notations. Let $K^{\epsilon} \in B(H_1, H_2)$ and $g^{\epsilon} \in H_2$ such that $\|K - K^{\epsilon}\|\ \le \epsilon$ and $\|g - g^{\epsilon}\|\ \le \epsilon$. Let $\{\phi_i\}_{i\in I}$ be an orthonormal basis for $M \subset H_1$. Let y_i's and ψ_i's be generated from the GGS(δ) on K relative M, and let y_i^{ϵ}'s and ψ_i^{ϵ}'s be generated from the GGS(δ) on K^{ϵ} relative to M. The symbols a_{ij}, b_i, A, and b are defined the same as before. Similarly, we define $a_{ij}^{\epsilon} = (K^{\epsilon}\phi_j, \psi_i^{\epsilon})$, $b_i^{\epsilon} = (g^{\epsilon}, \psi_i^{\epsilon})$, $A^{\epsilon} = (a_{ij}^{\epsilon})$, and $b^{\epsilon} = (b_i^{\epsilon})$. From the construction of the GGS(δ), the following Lemma 2.5 can be easily shown.

LEMMA 2.5. *If $\delta > 0$, then $a_{ii} = \begin{cases} \|y_i\| & \text{if } \|y_i\|\ \ge \delta \\ 0 & \text{if } \|y_i\|\ < \delta \end{cases}$ for each $i \in I$. Moreover, if $0 < \delta < \mu$, then $a_{ij} = 0$ for $i > j$.*

LEMMA 2.6. *Suppose $\delta > 0$ is chosen so that $\delta \neq\ \|y_i\|$ for all $i \in I$. Then, for each $i \in I$, $\lim_{\epsilon\to 0} y_i^{\epsilon} = y_i$ and $\lim_{\epsilon\to 0} \psi_i^{\epsilon} = \psi_i$.*

Proof. We proceed by mathematical induction on the index i. For

$i = 1$, $y_1 = K\phi_1$ and $y_1^\epsilon = K^\epsilon\phi_1$. Then

$$(1) \qquad \|y_1^\epsilon - y_1\| = \|K^\epsilon\phi_1 - K\phi_1\| \le \|K^\epsilon - K\| \le \epsilon.$$

From the inequality (1), $\lim_{\epsilon\to 0} y_1^\epsilon = y_1$. If $\|y_1\| < \delta$, then $\|y_1^\epsilon\| < \delta$ for sufficiently small ϵ and so $\psi_1^\epsilon = \psi_1 = 0$. Clearly, $\lim_{\epsilon\to 0}\psi_1^\epsilon = \psi_1$ for $\|y_1\| < \delta$. Since $\delta \ne \|y_1\|$, we consider the case $\|y_1\| > \delta$. Since $\lim_{\epsilon\to 0} y_1^\epsilon = y_1$, $\|y_1^\epsilon\| > \delta$ for sufficiently small ϵ and hence $\psi_1^\epsilon = \frac{y_1^\epsilon}{\|y_1^\epsilon\|}$. Note that $\|\psi_1 - \psi_1^\epsilon\| \le \frac{2}{\|y_1\|}\|y_1 - y_1^\epsilon\|$. Therefore, $\lim_{\epsilon\to 0}\psi_1^\epsilon = \psi_1$ for the case of $\|y_1\| > \delta$. Next, we suppose that $\lim_{\epsilon\to 0} y_i^\epsilon = y_i$ and $\lim_{\epsilon\to 0}\psi_i^\epsilon = \psi_i$ for all $i < k$. Then,

$$(2) \qquad \begin{aligned} \|y_k - y_k^\epsilon\| &= \|K\phi_k - \sum_{j=1}^{k-1}(K\phi_k, \psi_j)\psi_j - K^\epsilon\phi_k + \sum_{j=1}^{k-1}(K^\epsilon\phi_k, \psi_j^\epsilon)\psi_j^\epsilon\| \\ &\le \|K^\epsilon - K\| + \sum_{j=1}^{k-1}\|(K^\epsilon\phi_k, \psi_j^\epsilon)\psi_j^\epsilon - (K\phi_k, \psi_j)\psi_j\|. \end{aligned}$$

On the other hand, for each $j < k$,

$$(3) \qquad \begin{aligned} &\|(K^\epsilon\phi_k, \psi_j^\epsilon)\psi_j^\epsilon - (K\phi_k, \psi_j)\psi_j\| \\ &\le |(K^\epsilon\phi_k, \psi_j^\epsilon)|\|\psi_j^\epsilon - \psi_j\| + |(K^\epsilon\phi_k, \psi_j^\epsilon) - (K\phi_k, \psi_j)|\|\psi_j\| \\ &\le \|K^\epsilon\|\,\|\psi_j^\epsilon - \psi_j\| + |(\phi_k, (K^\epsilon)^*\psi_j^\epsilon - K^*\psi_j)| \\ &\le \|K^\epsilon\|\,\|\psi_j^\epsilon - \psi_j\| + \|(K^\epsilon)^*\psi_j^\epsilon - K^*\psi_j\| \\ &\le \|K^\epsilon\|\,\|\psi_j^\epsilon - \psi_j\| + \|(K^\epsilon)^*\|\,\|\psi_j^\epsilon - \psi_j\| + \|(K^\epsilon - K)^*\| \\ &= 2\|K^\epsilon\|\,\|\psi_j^\epsilon - \psi_j\| + \|K^\epsilon - K\|. \end{aligned}$$

Combining the inequalities (2) and (3), one obtains

$$(4) \qquad \begin{aligned} \|y_k - y_k^\epsilon\| &\le k\|K^\epsilon - K\| + 2\|K^\epsilon\|\sum_{j=1}^{k-1}\|\psi_j^\epsilon - \psi_j\| \\ &\le k\epsilon + 2(\|K\| + \epsilon)\sum_{j=1}^{k-1}\|\psi_j^\epsilon - \psi_j\|. \end{aligned}$$

By the induction hypothesis and the inequality (4), $\lim_{\epsilon\to 0} y_k^\epsilon = y_k$. If $\|y_k\| < \delta$, then $\|y_k^\epsilon\| < \delta$ for sufficiently small ϵ and hence $\psi_k = \psi_k^\epsilon = 0$. If $\|y_k\| > \delta$, then $\|y_k^\epsilon\| > \delta$ for sufficiently small ϵ and hence $\psi_k^\epsilon = \frac{y_k^\epsilon}{\|y_k^\epsilon\|}$. Since $\lim_{\epsilon\to 0} y_k^\epsilon = y_k$, $\lim_{\epsilon\to 0} \psi_k^\epsilon = \psi_k$ for the case of $\|y_k\| > \delta$.

LEMMA 2.7. *If $\delta > 0$ is chosen so that $\delta \neq \|y_i\|$ for all $i \in I$, then* $\lim_{\epsilon\to 0} \|A^\epsilon - A\|_2 = 0$.

Proof. Notice that $A = (a_{ij})$ and $A^\epsilon = (a_{ij}^\epsilon)$, where $a_{ij} = (K\phi_j, \psi_i)$ and $a_{ij}^\epsilon = (K^\epsilon\phi_j, \psi_i^\epsilon)$. Then,

$$
\begin{aligned}
|a_{ij} - a_{ij}^\epsilon| &= |(K\phi_j, \psi_i) - (K^\epsilon\phi_j, \psi_i^\epsilon)| \\
&= |(\phi_j, K^*\psi_i - (K^\epsilon)^*\psi_i^\epsilon)| \\
&\leq \|K^*\psi_i - (K^\epsilon)^*\psi_i^\epsilon\| \qquad (5)\\
&\leq \|K - K^\epsilon\| + \|K^\epsilon\|\,\|\psi_i - \psi_i^\epsilon\| \\
&\leq \epsilon + (\|K\| + \epsilon)\|\psi_i - \psi_i^\epsilon\|.
\end{aligned}
$$

Since $\|A - A^\epsilon\|_\infty = \max_{1\leq i\leq n} \sum_{j=1}^n |a_{ij} - a_{ij}^\epsilon|$, from the inequality (5)

$$\|A - A^\epsilon\|_\infty \leq n \max_{1\leq i\leq n} \left(\epsilon + (\|K\| + \epsilon)\|\psi_i - \psi_i^\epsilon\|\right).$$

Since $\lim_{\epsilon\to 0} \psi_i^\epsilon = \psi_i$ from Lemma 2.6 and $n < \infty$, $\lim_{\epsilon\to 0} \|A - A^\epsilon\|_\infty = 0$. Note that $\|A - A^\epsilon\|_2 \leq \sqrt{n}\|A - A^\epsilon\|_\infty$. Therefore, $\lim_{\epsilon\to 0} \|A - A^\epsilon\|_2 = 0$.

LEMMA 2.8. *If B is an $n \times n$ matrix and $\|E\|_2 < \frac{1}{\|B^\dagger\|_2}$, then rank$(B) \leq$ rank$(B + E)$.* (See [7, Theorem 8.15] for the proof.)

LEMMA 2.9. *If $0 < \delta < \mu$, rank$(A) =$ rank(A^ϵ) for sufficiently small ϵ.*

Proof. By Lemma 2.7, $\lim_{\epsilon\to 0} \|A - A^\epsilon\|_2 = 0$. Therefore, from Lemma 2.8, rank$(A) \leq$ rank(A^ϵ) for sufficiently small ϵ. Hence, if rank$(A) = n$, then rank$(A^\epsilon) = n$ for sufficiently small ϵ.

Suppose that rank$(A) < n$. It is easy to show that $a_{ii} = (K\phi_i, \psi_i) = (y_i, \psi_i) = 0$ if and only if $i \in I_0$, where $I_0 = \{i \in I \mid \|y_i\| < \delta\}$. Since

$0 < \delta < \mu$, rank(A) equals the cardinality of $I - I_0$. From Lemma 2.6, we have $\lim_{\epsilon\to 0} y_i^\epsilon = y_i$ for all $i \in I$. Hence, for each $i \in I_0$, $\|y_i^\epsilon\| < \delta$ when ϵ is sufficiently small. If we let $I_0^\epsilon = \{i \in I \mid \|y_i^\epsilon\| < \delta\}$, we see that $I_0 \subset I_0^\epsilon$ for sufficiently small ϵ. If $i \in I_0^\epsilon$, then $a_{ij}^\epsilon = (K^\epsilon \phi_j, \psi_i^\epsilon) = 0$ for all $j \in I$. Hence, rank(A^ϵ) is not greater than the cardinality of $I - I_0^\epsilon$. Since $I - I_0^\epsilon \subset I - I_0$ for sufficiently small ϵ, we must have rank$(A^\epsilon) \leq$ rank(A). Therefore, rank$(A^\epsilon) =$ rank(A) for sufficiently small ϵ.

LEMMA 2.10. *Let B and B^ϵ be $n \times n$ matrices, and let c and c^ϵ be vectors in R^n. Suppose that $w = B^\dagger c$ and $w^\epsilon = (B^\epsilon)^\dagger c^\epsilon$. If $\|B - B^\epsilon\|_2 < \frac{1}{\|B^\dagger\|_2}$ and rank$(B^\epsilon) \leq$ rank(B), then rank$(B^\epsilon) =$ rank(B) and*

$$\begin{aligned} \|w - w^\epsilon\|_2 \;\leq\; \|B^\dagger\|_2 \Bigg[& \frac{\|B - B^\epsilon\|_2 \; \|w\|_2 + \|c - c^\epsilon\|_2}{1 - \|B - B^\epsilon\|_2 \; \|B^\dagger\|_2} \\ & + \frac{\|B - B^\epsilon\|_2 \; \|B^\dagger\|_2 \; \|c - Bw\|_2}{1 - \|B - B^\epsilon\|_2 \; \|B^\dagger\|_2} + \|B - B^\epsilon\|_2 \; \|w\|_2 \Bigg]. \end{aligned}$$

(See [7, Theorem 9.7] for the proof.)

Proof of Theorem 2.4. Without the loss of generality, we can assume that $0 < \delta < \mu$. Let $z = A^\dagger b = (\beta_i)$ and $z^\epsilon = (A^\epsilon)^\dagger b^\epsilon = (\beta_i^\epsilon)$. Then, $f_M = \sum_{i=1}^n \beta_i \phi_i$ is the M-solution to $Kf = g$ and $f_M^\epsilon = \sum_{i=1}^n \beta_i^\epsilon \phi_i$ is the approximate M-solution to $K^\epsilon f = g^\epsilon$. Since $\{\varphi_i\}_{i\in I}$ is an orthonomal set, $\|f_M - f_M^\epsilon\| = \|z - z^\epsilon\|_2$. By Lemmas 2.7 and 2.9, $\lim_{\epsilon\to 0} \|A - A^\epsilon\|_2 = 0$ and rank$(A) =$ rank(A^ϵ) for sufficiently small ϵ. Therefore, applying Lemma 2.10, one obtains

$$\begin{aligned} \|f_M - f_M^\epsilon\| \;\leq\; \|A^\dagger\|_2 \Bigg[& \frac{\|A - A^\epsilon\|_2 \; \|z\|_2 + \|b - b^\epsilon\|_2}{1 - \|A - A^\epsilon\|_2 \; \|A^\dagger\|_2} \\ & + \frac{\|A - A^\epsilon\|_2 \; \|A^\dagger\|_2 \; \|b - Az\|_2}{1 - \|A - A^\epsilon\|_2 \; \|A^\dagger\|_2} + \|A - A^\epsilon\|_2 \; \|z\|_2 \Bigg] \end{aligned}$$

for sufficiently small ϵ. Since $\lim_{\epsilon\to 0} \|A - A^\epsilon\|_2 = 0$ and $\lim_{\epsilon\to 0} \|b - b^\epsilon\|_2 = 0$, we see that $\lim_{\epsilon\to 0} \|f_M - f_M^\epsilon\| = 0$. Hence, the GGS$(\delta)$ method provides the M-solution to $Kf = g$ which is well-posed under perturbations in both K and g.

EXAMPLE 2.11. Let $H_1 = H_2$, ϕ_1 and ϕ_2 be orthonormal vectors in H_1, $M = \langle \phi_1, \phi_2 \rangle$, and let P_i be the orthogonal projection of H_1 onto $\langle \phi_i \rangle$ for $i = 1, 2$. Define $K = P_1$ and $K^\epsilon = P_1 + \epsilon P_2$. Let $g = \alpha_1\phi_1 + \alpha_2\phi_2$ with $\alpha_2 \neq 0$ and $g^\epsilon = \alpha_1\phi_1 + (\alpha_2+\epsilon)\phi_2$. Then $\|K - K^\epsilon\| = \|g - g^\epsilon\| = \epsilon$, $K\phi_1 = \phi_1$, $K\phi_2 = 0$, $K^\epsilon\phi_1 = \phi_1$, and $K^\epsilon\phi_2 = \epsilon\phi_2$. Choose $\delta = \frac{1}{2}$. Then, it is easy to see that both the GGS(δ) and the GGS provide the exact M-solution $f_M = \alpha_1\phi_1$ of $Kf = g$. We now consider the perturbed problem $K^\epsilon f = g^\epsilon$. Using the GGS method, $y_1^\epsilon = \psi_1^\epsilon = \phi_1$, $y_2^\epsilon = \epsilon\phi_2$, and $\psi_2^\epsilon = \phi_2$ are generated, and so $f_M^\epsilon = \alpha_1\phi_1 + \left(1 + \frac{\alpha_2}{\epsilon}\right)\phi_2$. When ϵ converges to 0, f_M^ϵ never converges to f_M even if K^ϵ and g^ϵ converge to K and g respectively.

On the other hand, when using the GGS(δ) method, $y_1^\epsilon = \psi_1^\epsilon = \phi_1$, $y_2^\epsilon = \epsilon\phi_2$, and $\psi_2^\epsilon = \phi_2$ (if $\epsilon \geq \delta$) or 0 (if $\epsilon < \delta$) are generated. Hence, f_M^ϵ is equal to f_M for every ϵ which is less than δ. Therefore, the approximate M-solution f_M^ϵ for $K^\epsilon f = g^\epsilon$ obtained from the GGS(δ) method converges to the M-solution f_M of $Kf = g$ when ϵ converges to 0.

3. Numerical implementation of the GGS(δ)

In this section, it is assumed that $K : L_2[a,b] \to L_2[c,d]$ is an integral operator defined by $(Kf)(s) = \int_a^b k(s,t)f(t)dt$ with the kernel function $k(s,t) \in L_2([c,d] \times [a,b])$. The numerical method commonly used for integration of the form $\int_{-1}^1 f(x)dx$ is the Gauss-Legendre quadrature method. To this end, it is convenient to change variables so that we can work on $L_2[-1,1]$ instead of $L_2[a,b]$ and $L_2[c,d]$. First, we show how the change of variables should be handled for the problem of finding the M-solution of $(Kf)(s) = g(s)$ for all $s \in [c,d]$, where $g \in L_2[c,d]$.

Let $s = h_1(x) = \frac{c+d}{2} + \frac{d-c}{2}x$ and $t = h_2(y) = \frac{a+b}{2} + \frac{b-a}{2}y$, where $-1 \leq x, y \leq 1$. Put $k_0(x,y) = k(h_1(x), h_2(y))$, $G(x) = g(h_1(x))$, and $F(y) = f(h_2(y))$. Define an integral operator $K_0 : L_2[-1,1] \to L_2[-1,1]$ by $(K_0F)(x) = \int_{-1}^1 k_0(x,y)F(y)dy$ for $F \in L_2[-1,1]$ and $x \in [-1,1]$. Then, $(Kf)(s) = g(s)$ is equivalent to $(K_0F)(x) = \frac{2}{b-a}G(x)$. Suppose that N is a closed subspace of $L_2[a,b]$ with orthonormal basis $\{\phi_j(t)\}_{j\in J}$, where J is an index set which is at most countable. Define $\psi_j(y) = \phi_j(h_2(y))$ for each $j \in J$. Let N_0 be a subspace of $L_2[-1,1]$ which is the closed linear span of $\{\psi_j(y)\}_{j\in J}$. It is clear that $(\phi_i,\phi_j)_{[a,b]} = \delta_{ij} = \frac{b-a}{2}(\psi_i,\psi_j)_{[-1,1]}$, where δ_{ij} represents the Kronecker delta, and $(\cdot,\cdot)_{[a,b]}$ and $(\cdot,\cdot)_{[-1,1]}$ refer to the inner products on $L_2[a,b]$ and $L_2[-1,1]$ respectively. Using the above notations, we obtain the following.

LEMMA 3.1. *$g\in K(N)$ if and only if $G\in K_0(N_0)$. Moreover, $g\in \overline{K(N)}$ if and only if $G\in \overline{K_0(N_0)}$, where the bars denote the closure of the given subspaces.*

Let P be the orthogonal projection of $L_2[c,d]$ onto $\overline{K(N)}$. Define $P_0 : L_2[-1,1]\to L_2[-1,1]$ by $(P_0G)(x) = (Pg)(h_1(x))$, where $g = G\circ h_1^{-1}\in L_2[c,d]$. Then, it is easy to show that P_0 is the orthogonal projection of $L_2[-1,1]$ onto $\overline{K_0(N_0)}$. Since $Kf = Pg$ is equivalent to $K_0F = P_0(\frac{2}{b-a}G)$ and $(f,f)_{[a,b]} = \frac{b-a}{2}(F,F)_{[-1,1]}$, the following theorem is immediately obtained.

THEOREM 3.2. *$f_N(t)$ is the N-solution of $Kf = g$ if and only if $F_{N_0}(y) = (f_N\circ h_2)(y)$ is the N_0-solution of $K_0F = \frac{2}{b-a}G$.*

By taking $N = L_2[a.b]$ in Theorem 3.2, we obtain the following.

COROLLARY 3.3. *$f_0(t)$ is the LSSMN of $Kf = g$ if and only if $F_0(y) = (f_0\circ h_2)(y)$ is the LSSMN of $K_0F = \frac{2}{b-a}G$.*

Let's now consider the numerical implementation of the GGS(δ) for approximating the M-solution of the form

$$(Kf)(s) = \int_0^1 k(s,t)f(t)dt = g(s), \quad 0\le s\le 1.$$

First, we will transform it to the equivalent problem

$$(K_0F)(x) = \int_{-1}^1 k_0(x,y)F(y)dy = 2G(x), \quad -1\le x\le 1,$$

by using the change of variables $s = \frac{1+x}{2}$ and $t = \frac{1+y}{2}$, where $k_0(x,y) = k(\frac{1+x}{2},\frac{1+y}{2})$, $F(y) = f(\frac{1+y}{2})$, and $G(x) = g(\frac{1+x}{2})$. Next, we will choose an n dimensional subspace $(M_0)_n$ of $L_2[-1,1]$ so that $(M_0)_n = \langle\psi_1(y), \dots,\psi_n(y)\rangle$, where $\{\psi_i(y)\}_{i=1}^n$ is an appropriate orthonormal set in

$L_2[-1,1]$. As a typical example for such an orthonormal set in $L_2[-1,1]$, the normalized Legendre polynomials can be chosen. Then, the $(M_0)_n$-solution $F_n(y)$ of the transformed problem $K_0F = 2G$ is approximated using the GGS(δ) method. Let $\hat{F}_n(y)$ denote the approximate $(M_0)_n$-solution to $K_0F = 2G$ obtained from the GGS(δ). Note that from Theorem 3.2, $f_n(t) = F_n(2t-1)$ is the M_n-solution of the given problem $Kf = g$, where $M_n = \langle \phi_1(t), \dots, \phi_n(t) \rangle$ and $\phi_i(t) = \psi_i(2t-1)$. Thus, the approximate M_n-solution to $Kf = g$ computed from the GGS(δ) is given by $\hat{f}_n(t) = \hat{F}_n(2t-1)$.

All computings were done in Fortran using double precision arithmetic which means about fifteen decimal digits of accuracy. The number δ for the GGS(δ) method was taken to be 10^{-13}. The second step of the GGS(δ) requires the computation of the minimum norm solution of $Ax = b$. If A has full rank, then A is an upper triangular matrix and so the minimum norm solution of $Ax = b$ is solved by the Linpack subroutine DTRSL [3, Chapter 6] which solves a linear system with triangular matrix. If A does not have full rank, then the minimum norm solution of $Ax = b$ is solved using the Linpack subroutine DSVDC [3, Chapter 11] which carrys out the singular value decomposition of a matrix. The decision of whether or not A has full rank was made in the following way: If $\|y_i\| < \delta$ for an $i \in I$, then all elements of the i-th row of A consist of zeros and hence A does not have full rank. If $\|y_i\| \geq \delta$ for every $i \in I$, then A has full rank.

The DSVDC supplies $\{\sigma_1, \dots, \sigma_n\}$, the singular values of A. In order to decide which of σ_i's are nonzero, the same tolerance δ mentioned above is used. In other words, if $\sigma_i < \delta$, then σ_i is set to zero. All inner products required during the GGS(δ) were done using the repeated 4-point Gauss-Legendre quadrature method. The examples tested in this paper are described below. For each example, f_0 denotes the LSSMN to $Kf = g$, and $(M_0)_n = \langle \psi_1(y), \dots, \psi_n(y) \rangle$ is chosen as an n-dimensional subspace of $L_2[-1,1]$, where $\psi_i(y)$ is the normalized Legendre polynomial of degree $(i-1)$. The exact M_n-solution f_n is also provided in each of the examples below if possible.

EXAMPLE 3.4. Let $k(s,t) = s + t$ and $g(s) = s$. Then $f_0(t) = 4 - 6t$, and $f_n(t) = f_0(t)$ for $n \geq 2$.

EXAMPLE 3.5. Let $k(s,t) = \cos(st)$ and $g(s) = \frac{\sin s}{s} + \frac{\cos s - 1}{s^2}$. Then $f_0(t) = t$, and $f_n(t) = f_0(t)$ for $n \geq 2$.

EXAMPLE 3.6. Let $k(s,t) = (s-t)^2$ and $g(s) = s^2 - \frac{2}{3}s + \frac{1}{4}$. Then $f_0(t) = 15t^2 - 17t + \frac{9}{2}$, $f_2(t) = \frac{29}{12} - 2t$, and $f_n(t) = f_0(t)$ for $n \geq 3$.

EXAMPLE 3.7. Let $k(s,t) = \sin 2(s+t-1)$ and $g(s) = 5\sin(2s-1) + 3\cos(2s-1)$. Then $f_0(t) = \left(\frac{10}{2+\sin 2}\right)\cos(2t-1) + \left(\frac{6}{2-\sin 2}\right)\sin(2t-1)$, $f_2(t) = \frac{5}{2\sin 1} + \frac{3}{2(\sin 1 - \cos 1)}(2t-1)$, and so on.

The numerical results for the above examples are summarized in Table 1. $\|f_n - \hat{f}_n\|$ is computed to see how close the approximate M_n-solution $\hat{f}_n$ obtained from the GGS(δ) is to the exact M_n-solution f_n, and $\|f_0 - \hat{f}_n\|$ is computed to see how well the $\hat{f}_n$ approximates f_0. Since the exact M_n-solution f_n for Example 3.7 is not available by hand calculation if $n > 3$, the value of $\|f_n - \hat{f}_n\|$ is not listed in Table 1 for $n = 4$ and 5. The numbers in Table 1 under the column $\|f_n - \hat{f}_n\|$ range from 10^{-16} to 10^{-13}, which shows numerically that the GGS(δ) is a well-posed method for finding the M-solution to $Kf = g$. As can be seen from Table 1, for Examples 3.4 to 3.6 $\hat{f}_n$ converges very fast to f_0, whereas for Example 3.7 $\hat{f}_n$ converges very slowly to f_0. This is because we cannot approximate $\sin y$ and $\cos y$ very accurately using low-degree Legendre polynomials.

If we choose $(M_0)_2 = \langle \sin y, \cos y \rangle$ for Example 3.7, then it is easy to see that $f_2(t)$ equals $f_0(t)$. For this choice of $(M_0)_2$, the computed value of $\|f_0 - \hat{f}_2\| = \|f_2 - \hat{f}_2\|$ is about 10^{-14}, i.e., the computed M_2-solution $\hat{f}_2$ is a good approximation to f_0 as well as f_2. Hence, in order for the computed M_n-solution $\hat{f}_n$ to approximate f_0 well, it is natural to choose M_n to reflect a priori knowledge of the f_0. Since the goal of this paper is to develop a well-posed method for finding the M-solution to $Kf = g$, the problem of choosing a suitable subspace M_n so that f_n can approximate f_0 well is not considered herein and it will be studied in the future work.

Table 1. Numerical results for the GGS(δ)

Example	n	$\|\|f_n - \bar{f}_n\|\|$	$\|\|f_0 - \bar{f}_n\|\|$
3.4	2	5.10×10^{-15}	5.10×10^{-15}
	3	5.89×10^{-15}	5.89×10^{-15}
	4	5.89×10^{-15}	5.89×10^{-15}
3.5	2	1.09×10^{-13}	1.09×10^{-13}
	3	1.09×10^{-13}	1.09×10^{-13}
	4	1.09×10^{-15}	1.09×10^{-15}
3.6	2	3.74×10^{-16}	1.1932
	3	1.96×10^{-15}	1.96×10^{-15}
	4	1.96×10^{-15}	1.96×10^{-15}
3.7	2	6.53×10^{-14}	7.08×10^{-1}
	3	1.98×10^{-13}	5.75×10^{-1}
	4		5.54×10^{-1}
	5		5.53×10^{-1}

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