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RELATIONS BETWEEN THE ITO PROCESSES

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1. Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space with $\mathcal{F}$ a σ-algebra of subsets of the measure space Ω and P a probability measure on Ω. Suppose $a > 0$ and let $(\mathcal{F}_t)_{t\in[0,a]}$ be an increasing family of sub-σ- algebras of $\mathcal{F}$. If $r > 0$, let $J = [-r, 0]$ and $C(J, \mathbf{R}^n)$ the Banach space of all continuous paths $\gamma : J \to \mathbf{R}^n$ with the sup-norm $||\gamma|| = \sup_{s\in J} |\gamma(s)|$ where $|\cdot|$ denotes the Euclidean norm on $\mathbf{R}^n$. Let E, F be separable real Banach spaces and $L(E, F)$ be the Banach space of all continuous linear maps $T : E \to F$. Throughout this paper, we restrict ourselves to a class of the autonomous stochastic functional differential equations

$$x(\omega)(t) = x(\omega)(0) + \int_0^t H_1(x(u)(\omega))du + (\omega)\int_0^t G_1(x(u)(\cdot))dw_1(\cdot)(u)$$

$$\begin{aligned} y'(\omega)(t) = y'(\omega)(0) &+ \int_0^t H_2(y(u)(\omega), y'(u)(\omega))du \\ &+ (\omega)\int_0^t G_2(y(u)(\cdot), y'(u)(\cdot))dw_2(\cdot)(u) \end{aligned}$$

in which the coefficient processes factors through $H_1 : C(J, \mathbf{R}^n) \to \mathbf{R}^n$, $H_2 : C(J, \mathbf{R}^n) \times C(J, \mathbf{R}^n) \to \mathbf{R}^n$ and $G_1 : C(J, \mathbf{R}^n) \to L(\mathbf{R}^m, \mathbf{R}^n)$, $G_2 : C(J, \mathbf{R}^n) \times C(J, \mathbf{R}^n) \to L(\mathbf{R}^m, \mathbf{R}^n)$, while the noise processes takes the form $\{t + w_i(t) : t \in [0, a], i = 1, 2\}$ with w_i an m-dimensional Brownian motions on a filtered probability space. We also make the following standing hypotheses:

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Hypotheses (1) The coefficients H_i, $G_i (i = 1, 2)$ are globally Lipschitz with Lipschitz constants $L_i (i = 1, 2)$, respectively.

Hypotheses (2) The coefficients are continuous.

In this note, we observe the relations between two Ito processes in view of the consecutive result of [1] and [2]. We examine the relations of two processes using the linear and bilinear maps in detail.

2. The main results

We begin with:

THEOREM 1. *Suppose the Hypotheses* (1), (2) *are satisfied and* $E(y'(0))$ *is bounded. Then* $\|\frac{1}{t}E(x(t) - x(0))\|$ *and* $\|\frac{1}{t}E(y(t) - y(0))\|$ *are bounded and*

$$\lim_{t \to 0+} \frac{1}{t} E(x(t) - x(0)) = H_1(x(0)),$$

$$\lim_{t \to 0+} \frac{1}{t} E(y(t) - y(0)) = E(y'(0)).$$

Proof. Now

$$x(\omega)(t) - x(\omega)(0) = \int_0^t H_1(x(u)(\omega))du + (\omega) \int_0^t G_1(x(u)(\cdot))dw_1(\cdot)(u)$$

and

$$\begin{aligned} y(\omega)(t) - y(\omega)(0) = ty'(\omega)(0) &+ \int_0^t (t-u) H_2(y(u)(\omega), y'(u)(\omega))du \\ &+ (\omega) \int_0^t (t-u) G_2(y(u)(\cdot), y'(u)(\cdot))dw_2(\cdot)(u) \end{aligned}$$

for each $t \in [0, a]$, a.a. $\omega \in \Omega$. By the martingale property of the Ito integral([3, II.2], [4]), it follows that

$$E\Big(\frac{1}{t}(x(t) - x(0))\Big) = \frac{1}{t} \int_0^t E(H_1(x(u)))du$$

and

$$E\Big(\frac{1}{t}(y(t)-y(0))\Big) = E(y'(0)) + \frac{1}{t}\int_0^t E((t-u)H_2(y(u),y'(u)))du.$$

Hence using the Leibniz's formula, we have

$$\lim_{t\to 0+} E\Big(\frac{1}{t}(x(t)-x(0))\Big) = \lim_{t\to 0+}\frac{1}{t}\int_0^t E(H_1(x(u)))du = H_1(x(0))$$

and

$$\begin{aligned}&\lim_{t\to 0+} E\Big(\frac{1}{t}(y(t)-y(0))\Big)\\ &= E\left(y'(0)) + \lim_{t\to 0+}\frac{1}{t}\int_0^t E((t-u)H_2(y(u),y'(u))\right)du\\ &=E(y'(0)).\end{aligned}$$

On the other hand, let K_1, $K_2 > 0$ be such that $|H_1(x(0))| \le K_1$, $|H_2(y(0),y'(0))| \le K_2$ for all $x(0)$, $y(0)$ and $y'(0)$. Clearly

$$|\frac{1}{t}E(x(t)-x(0))| \le \frac{1}{t}\int_0^t |E[H_1(x(t))]|du \le K_1$$

and

$$\begin{aligned}\left|\frac{1}{t}E(y(t)-y(0))\right| &\le E(|y'(0)|) + \frac{1}{t}\int_0^t |E[H_2(y(t),y'(t))]|\,du\\ &\le E(y'(0)) + K_2.\end{aligned}$$

Therefore $||\frac{1}{t}E(x(t)-x(0))||$ and $||\frac{1}{t}E(y(t)-y(0))||$ are bounded.

COROLLARY 2. *Let f be a continuous linear map on $C(J,\mathbf{R}^n)$. Then*

$$\begin{aligned}\lim_{t\to 0+}\frac{1}{t}Ef(x(t)-x(0)) = f(H_1(x(0))),\ \lim_{t\to 0+}\frac{1}{t}Ef(y(t)-y(0))\\ = f(E(y'(0))).\end{aligned}$$

Proof. This results follow from the fact that

$$\frac{1}{t}Ef(x(t)-x(0)) = f\left[\frac{1}{t}E(x(t)-x(0))\right],$$
$$\frac{1}{t}Ef(y(t)-y(0)) = f\left[\frac{1}{t}E(y(t)-y(0))\right].$$

For each $t>0$ and $\omega\in\Omega$, define $z_i(\omega)(t)$ by

$$z_i(\omega)(t) = \frac{1}{\sqrt{t}}[w_i(\omega)(t)-w_i(\omega)(0)],\ (i=1,2)$$

Then we now meet:

THEOREM 3. *Suppose Hypotheses* (1), (2) *are satisfied. Then*

$$\lim_{t\to 0+} E\left\|\frac{1}{\sqrt{t}}(x(t)-x(0)) - G_1(x(0))z_1(t)\right\|^2 = 0$$

and

$$\lim_{t\to 0+} E\left\|\frac{1}{\sqrt{t}}(y(t)-y(0)) - G_2(y(0),y'(0))z_2(t)\right\|^2 = 0.$$

Proof. Writing that

$$\begin{aligned}&\frac{1}{\sqrt{t}}(x(t)-x(0)) - G_1(x(0))z_1(t)\\ &= \frac{1}{\sqrt{t}}\int_0^t H_1(x(u))du + \frac{1}{\sqrt{t}}\int_0^t G_1(x(u))dw_1(u)\\ &\quad - G_1(x(0))\left[\frac{1}{\sqrt{t}}\int_0^t dw_1(u)\right]\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{\sqrt{t}}(y(t)-y(0)) - G_2(y(0),y'(0))z_2(t)\\ &= \sqrt{t}y'(0) + \frac{1}{\sqrt{t}}\int_0^t H_2(y(u),y'(u))du\\ &\quad + \frac{1}{\sqrt{t}}\int_0^t G_2(y(u),y'(u))dw_2(u) - G_2(y(0),y'(0))\left[\frac{1}{\sqrt{t}}\int_0^t dw_2(u)\right],\end{aligned}$$

we have

$$
\begin{aligned}
&E\sup_{s\in J}\left|\frac{1}{\sqrt{t}}(x(t)-x(0))(s)-G_1(x(0))z_1(t)(s)\right|^2\\
&\le 2\frac{1}{t}E\sup_{s\in J}\left|\int_0^t H_1(x(u))du\right|^2\\
(1)\qquad &+2\frac{1}{t}E\sup_{s\in J}\left|\int_0^t [G_1(x(u))-G_1(x(0))]dw_1(u)\right|^2\\
&\le \frac{2}{t}t\int_0^t E|H_1(x(u))|^2du+\frac{2M}{t}\int_0^t E||G_1(x(u))-G_1(x(0))||^2du
\end{aligned}
$$

and

$$
\begin{aligned}
&E\sup_{s\in J}\left|\frac{1}{\sqrt{t}}(y(t)-y(0))(s)-G_2(y(0),y'(0))z_2(t)(s)\right|^2\\
(2)\qquad &\le 2E(t|y'(0)|^2)+4\int_0^t E|H_2(y(u),y'(u))|^2du\\
&+\frac{4N}{t}\int_0^t E||G_2(y(u),y'(u))-G_2(y(0),y'(0))||^2du
\end{aligned}
$$

for some constants $M,N>0$.
But it follows from the Hypotheses (1) that

$$E|H_1(x(t))|^2\le 2|H_1(x(0))|^2,\ E|H_2(y(t),y'(t))|^2\le 2|H_2(y(0),y'(0))|^2$$

and

$$
\begin{aligned}
&\lim_{t\to 0+}E||G_1(x(t))-G_1(x(0))||^2=0,\\
&\lim_{t\to 0+}E||G_2(y(t),y'(t))-G_2(y(0),y'(0))||^2=0.
\end{aligned}
$$

Therefore letting $t\to 0+$ in (1) and (2), we obtain the results.

We conclude with:

COROLLARY 4. *Let g be a continuous bilinear form on $C(J, \mathbf{R}^n)$. Then*

$$\lim_{t\to 0+}\Big[\frac{1}{t}Eg(x(t)-x(0), y(t)-y(0)) - Eg(G_1(x(0))z_1(t), G_2(y(0), y'(0))z_2(t))\Big] = 0.$$

Proof. Since g is bilinear, we write

$$\begin{aligned}
&\frac{1}{t}g(x(t)-x(0), y(t)-y(0)) - g(G_1(x(0))z_1(t), G_2(y(0), y'(0))z_2(t))\\
&= g\Big(\frac{1}{\sqrt{t}}(x(t)-x(0)\Big) - G_1(x(0))z_1(t),\\
&\quad \frac{1}{\sqrt{t}}(y(t)-y(0)) - G_2(y(0), y'(0))z_2(t))\\
&\quad + g\Big(\frac{1}{\sqrt{t}}(x(t)-x(0)\Big) - G_1(x(0))z_1(t), G_2(y(0), y'(0))z_2(t))\\
&\quad + g(G_1(x(0))z_1(t), \frac{1}{\sqrt{t}}(y(t)-y(0)) - G_2(y(0), y'(0))z_2(t)).
\end{aligned}$$

By the continuity of g, we obtain

(3)
$$\begin{aligned}
&\Big|\frac{1}{t}Eg(x(t)-x(0), y(t)-y(0)) - g(G_1(x(0))z_1(t), G_2(y(0), y'(0))z_2(t))\Big|\\
&\le \|g\|E\Big[\Big\|\frac{1}{\sqrt{t}}(x(t)-x(0)) - G_1(x(0))z_1(t)\Big\|\\
&\quad \cdot\Big\|\frac{1}{\sqrt{t}}(y(t)-y(0)) - G_2(y(0), y'(0))z_2(t))\Big\|\Big]\\
&\quad + \|g\|\Big[E\Big\|\frac{1}{\sqrt{t}}(x(t)-x(0)) - G_1(x(0))z_1(t)\Big\|^2\Big]^{\frac{1}{2}}\left[E\|G_2(y(0), y'(0))z_2(t)\|^2\right]^{\frac{1}{2}}\\
&\quad + \|g\|\Big[E\Big\|\frac{1}{\sqrt{t}}(y(t)-y(0)) - G_2(y(0), y'(0))z_2(t)\Big\|^2\Big]^{\frac{1}{2}}\left[E\|G_1(x(0))z_1(t)\|^2\right]^{\frac{1}{2}}.
\end{aligned}$$

But it follows that

$$\begin{aligned}
E\|G_1(x(0))z_1(t)\|^2 &\le \frac{1}{t}E\sup_{s\in[-t,0]}|w_1(t)-w_1(0)|^2(s)\|G_1(x(0)\|^2\\
&\le \|G_1(x(0))\|^2
\end{aligned}$$

and

$$E||G_2(y(0), y'(0))z_2(t)||^2$$
$$\leq \frac{1}{t} E \sup_{s \in [-t,0]} |w_2(t) - w_2(0)|^2(s)||G_2(y(0), y'(0))||^2 \leq ||G_2(y(0), y'(0))||^2.$$

Therefore letting $t \to 0+$ in (3), the result follows from the Theorem 3.

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