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VOLUME PROBLEMS ON LORENTZIAN MANIFOLDS

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1. Introduction

Inspired in [2,9,10,17], P. E. Ehrlich and S. B. Kim in [4] cultivated the Riccati equation related to the Raychaudhuri equation of General Relativity for the stable Jacobi tensor along the geodesics to extend the Hawking-Penrose conjugacy theorem to

$$f(t) = Ric(c(t)', c'(t)) + tr(\sigma(A)^2)$$

where $\sigma(A)$ is the shear tensor of A for the stable Jacobi tensor A with $A(t_0) = Id$ along the complete Riemannian or complete nonspacelike geodesics c.

On the other hand, when we investigate some properties of local distance functions on Lorentzian manifolds of positive Ricci curvature, we need a proper concept of "completeness" as usually given in Riemannian manifolds. There is a similar concept called "geodesic completeness". However, it is not helpful because the Hopf-Rinow theorem does not hold on Lorentzian manifolds. Moreover, it is well-known that the positive Ricci curvature bounded away from zero on all timelike vectors tends to produce nonspacelike geodesic incompletenes, which is a kind of singularity theorem on space-times, cf. Theorem 11.41 in [1]. Thus, we give the hypothesis called "global hyperbolicity" to the Lorntzian manifolds in this paper since the global hyperbolicity guarantees the existance of a timelike maximal geodesic segment between any pair of chronological related points, cf. [1].

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There are some papers in Riemannian geometry for the local distance functions related to the Riccati equations given by Eschenberg, Karcher, and Meyer [5,11,14]. To establish the theory for the local distance functions on Lorentzian manifolds, we start with the Jacobi tensor along the timelike geodesic starting from a point of the given globally hyperbolic space-time.

Further, the volume problem on Lorentzian or semi-Riemannian manifolds has been an undeveloped area since it is hard to define the geometric volume of any compact set even O'Neill in [15] defined the volume element on an oriented semi-Riemannian manifold, and since Shimming and Matel-Kaminska in [16] obtained some results of the volume problem where they have just tried less geometrically intrinsic objects called exponentiated "truncated light cones" on Lorentzian or semi-Riemannian manifolds. Moreover, we can not investigate the local distance functions for null geodesics since any distance along null geodesics is zero.

Thus we construct a compact set given by a geodesic wedge contained in the chronological future $I^+(p)$ for some point p in a globally hyperbolic space-time M, and then we define the volume of the geodesic wedge to investigate some volume problems.

2. Preliminaries

Let $c : (a,b) \to (M,g)$ denote unit speed timelike geodesic with $g(c',c') = -1$ in a Lorentzian manifold (M,g) and let $N(c(t))$ denote the $(n-1)$-dimensional subspace of $T_{c(t)}M$ consisting of tangent vectors orthogonal to $c'(t)$. A (1,1)-tensor field $A(t)$ on $V^{\perp}(c)$ is a linear map $A = A(t) : N(c(t)) \to N(c(t))$ for each t in (a,b). For the orthonormal basis of parallel fields $\{P_j\}$ for $N(c)$ let $A(P_i) = \sum \varphi_{ij}(t)P_j$. Then the covariant derivative $A'(P_i) = \sum \varphi'_{ij}(t)P_j$. And $RA(t)(v) = R(A(t)(v), c'(t))c'(t)$ where R is the curvature tensor field on $N(c)$ and the adjoint A^* is given by $g(A^*(t)(v), w) = g(A(t)w, v)$ for all $v, w \in N(c(t))$.

A smooth (1,1) tensor field $A(t)$ which satisfies $A'' + RA = 0$ and $\ker(A(t)) \cap \ker(A'(t)) = \{0\}$ for all t in (a,b) is called a *Jacobi tensor field* and the Jacobi tensor A is called a *Lagrange tensor* if it satisfies the further condition $(A')^*A - A^*A' = 0$, which is satisfied if $A(t) = 0$ for some t in (a,b).

We may choose two different kinds of Jacobi tensors to study a level surface $f^{-1}(t_0)$ along a local distance function $f : M \to \mathbf{R}$. The first is the smooth Jacobi tensor field A along c satisfying the initial conditions $A(t_0) = 0, A'(t_0) = id$ which is called the *conjugate Jacobi tensor*. Then if $v \in \ker(A(t_1))$ and P is the parallel field along c with $P(c(t_1)) = v$, the vector field $J := A(P)$ is a nontrivial Jacobi field along c with $J(t_0) = J(t_1) = 0$. In this case, we have $|tr(A'A^{-1}(t_0))| = +\infty$, This first has been given in [7] for Riemannian or timelike geodesics and in [12,13] for nonspacelike geodesics in Lorentzian, and, more generally, in pseudo-Riemannian manifolds.

We now consider a second Jacobi tensor along c which is assumed that c is free of conjugate points. Then the *stable Jacobi tensor field* A along c is constructed as a limit as $s \to +\infty$ of Jacobi tensors D_s satisfying the boundary value conditions $D_s(t_0) = id, D_s(s) = 0$, but $D_s'(s) \neq 0$, cf. [3,6,8]. In this case, we may assume that c is defined on $(-\infty, +\infty)$. It is very useful to the conjugate point along c in the indirect method as in [4].

In both of these methods, however, the technique of passing from the Jacobi equation $A'' + RA = 0$ to the associated Riccati inequality (in Riemannian geometry) or Riccati equation (in General Relativity) for the trace of the tensor $B := A'A^{-1}$ defined at points where A is nonsingular has been found to be useful.

Let $A(t)$ then, be a Lagrange tensor along the nonspacelike or Riemannian geodesic $c : (a, b) \to (M, g)$. Except at the isolated parameter values t where $\det A(t) = 0$, the self adjoint tensor $B = A'A^{-1}$ may be formed. Then the *expansion* $\theta(A)$ and *shear tensor* $\sigma(A)$ of A are defined by

$$\theta = \theta(A) := tr(B), \quad \sigma = \sigma(A) := B - a\theta(A)Id$$

where $a = 1/(n-1)$ if c is Riemannian or timelike. Then at points where $\det A(t) \neq 0$, as Lemma 1 in [7] one has

$$\theta = tr(A'A^{-1}) = (\det(A))'/\det(A). \tag{2.1}$$

Thus for a Jacobi tensor A with $A(t_0) = 0, A'(t_0) = Id, t = t_1 \neq 0$ is conjugate to t_0 if $|\theta(t)| \to +\infty$ as $t \to t_1$. Note also that since $\sigma(A)$ is self-adjoint, $tr(\sigma^2) \geq 0$ and equality holds at some t iff $\sigma(A)(t) = 0$.

Moreover, $B' = (A'A^{-1})'$ yields

$$B' = A''A^{-1} - A'A^{-1}A'A^{-1} = -R(.,c')c' - B \circ B, \tag{2.2}$$

a Riccati-type equation for B. Then using identities $B = \sigma(A) - a\theta(A)Id$ and $\theta' = (tr(B))' = tr(B') = -tr(R) - tr(B^2) = -Ric(c', c') - tr(B^2)$, one obtains the so-called Raychaudhuri equation of General Relativity

$$\theta' + a\theta^2 + (Ric(c', c') + tr(\sigma(A)^2)) = 0. \tag{2.3}$$

Thus starting with the Jacobi equation $A'' + RA = 0$, forming $B = A'A^{-1}$, we have obtained a Riccati equation $\theta' + a\theta^2 + f_A = 0$ for $\theta(A) = tr(B)$ where

$$f(t) = f_A(t) = Ric(c'(t), c'(t)) + tr((\sigma(A)(t))^2). \tag{2.4}$$

Now, let M^n be a globally hyperbolic space-time with $Ric(v, v) \geq (n-1)k > 0$ for any unit timelike vecots v and for some $k > 0$.

For $p \in M$, let $Fut(T_pM)$ be a set of all future directed timelike vectors v in T_pM such that $\exp_p v$ exists, and set $cut_\nu(p)$ be the shortest length from p to the cut point of p in the direction ν. Then we may have the distance function $f : \exp_p(Fut(T_pM)) \to \mathbf{R}$ given by $f(q) = d(p, q)$. For $t_0 \in f(\exp_p(Fut(T_pM)))$, set $E := \{v \in Fut(T_pM) | < v, v >= -1,\ \exp_p tv$ is defined for $0 \leq t \leq t_0 + \epsilon < cut_v(p)$ for some $\epsilon > 0\}$.

For any $v \in E$, we have a radial geodesic $\gamma_v(t) = \exp_p tv$ with $\gamma_v(0) = p,\ \gamma_v'(0) = v$. Then we have a normal variation α of $\gamma_v(t) = \exp_p tv$ given by

$$\alpha(t, s) = \exp_p t(v + sw) \tag{2.5}$$

with $\alpha_{*|(t_0,0)}(\frac{d}{ds}) = D_{t_0 v} \exp_p(t_0 w)$ for any $w \in T_{\gamma_v(t_0)} f^{-1}(t_0)$.

In this paper, we don't use the stable Jacobi tensor along γ_v starting from the level set $f^{-1}(t_0)$ since we can not find a stable Jacobi tensor in the space-time of constant curvature $k > 0$. For, we don't have any solution $\varphi_\infty(t) = \lim_{s \to \infty} \varphi_s(t)$ of the equation $\varphi_s'' + k\varphi_s = 0$ with $\varphi_s(0) = 1,\ \varphi_s'(s) = 0$. Instead, we will compare the Riccati equations arising between the space-time of the Ricci curvature and the space-time of constant curvature. Hence, we may choose a conjugate Jacobi tensor field A along γ_v satisfying $A(0) = 0$, $A'(0) = Id$.

Let $D : T_{\gamma_v(t_0)} f^{-1}(t_0) \to T_{\gamma_v(t_0)} f^{-1}(t_0)$ be any invertible linear transformation and suppose P is a parallel field in $\gamma_v(t))^{\perp}$ with $P(t_0) = w$ for $w \in T_{\gamma_v(t_0)} f^{-1}(t_0)$, $A(t_0) = D$. Then

$$(2.6) \qquad (AP)(t) = A(t)P(t) = \alpha_{*|(t,0)}(AP)(t_0) = \alpha_{*|(t,0)}(Dw).$$

Clearly, $J = AP$ is a Jacobi field along γ_v. Since D is invertible, the equation (2.6) induces a unique variation of $\gamma_v(t)$ with $\alpha(t_0, 0) = \gamma_v(t_0)$. Moreover, the conjugate Jacobi tensor A along γ_v may be assumed $A(t) \neq 0$ for $t \in (t_0, cut_v(p))$.

Now, $f^{-1}(t_0)$ is the spacelike hypersurface since its $grad(f)$ is timelike i.e., $< (grad(f), grad(f) >= -1$. Then, as is well known, the integral curves of $grad(f)$ are unit speed maximal geodesics and the level submanifolds $\{f^{-1}(t_0)\}$ are smooth hypersurfaces with $N := grad(f)$ serving as a unit normal field. Given N on M, define the shape operator

$$L : T_p(f^{-1}(t_0)) \to T_p(f^{-1}(t_0))$$

by $L(v) = -\nabla_v N$ in $T_p M$, i.e., $L = -\nabla N$. Since, $A'P = (AP)' = \nabla_N J = \nabla_J N = \nabla N(AP)$, $A' = \nabla N \circ A$, and thus, $A'A^{-1} = \nabla N = -L$. Since L is self-adjoint, the Wronskian $W(A, A) = (A')^* A - A^* A' = 0$. Thus, A is a Lagrange tensor. Putting $B = A'A^{-1}$, we have the expansion tensor θ and the shear tensor σ of A.

Note that, setting $E(t) = \dfrac{A(t)}{t}$ for $t \neq 0$, using $A'(0) = Id$, we have $\lim_{t \to 0} E(t) = Id$, and $\lim_{t \to 0} E'(t) = 0$. Further, $\lim_{t \to 0} tB(t) = Id$, and $\lim_{t \to 0} (tB(t))' = 0$. Hence, $tB(t) = Id + O(t^2)$, and thus,

$$(2.7). \qquad \sigma(0) = 0$$

(cf. Lemma 3 in [7]). As (2.3), one obtains the so-called Raychaudhuri equation of General Relativity

$$(2.8) \qquad \theta' + \frac{1}{n-1}\theta^2 + Ric(\gamma_v', \gamma_v') + tr(\sigma^2) = 0.$$

on $(0, cut_v(p))$. Putting $b = \dfrac{\theta}{n-1}$,

$$b' + b^2 + \frac{Ric(\gamma_v', \gamma_v')}{n-1} + \frac{tr(\sigma^2)}{n-1} = 0. \tag{2.9}$$

Moreover, $A(0) = 0$ implies $|b(0)| = +\infty$. Further, since $tr(\sigma^2) \geq 0$, using the Ricci curvature condition, we obtain the inequality

$$b' + b^2 + k \leq 0. \tag{2.10}$$

Now, let Q_k be the space-time of constant curvature $k > 0$. Then we have a conjugate Jacobi tensor A along any geodesic with $A(0) = 0$, $A'(0) = Id$ which induces a Riccati equation as follows;

At first, we have the solution $s(t) = \dfrac{1}{\sqrt{k}} \sin \sqrt{k}t$ satisfying the equation $s'' + ks = 0$ with $s(0) = 0$, $s'(0) = 1$. Hence, $\dfrac{s'(t)}{s(t)} = \sqrt{k} \cot \sqrt{k}t$. Thus, we have the conjugate Jacobi tensor A with $A(0) = 0$, $A'(0) = Id$.

Moreover, $B = A'A^{-1} = \sqrt{k} \cot \sqrt{k}tId$, $\theta = tr(B) = n\sqrt{k} \cot \sqrt{k}t$, and $\sigma = B - \dfrac{\theta}{n} Id = 0$. Thus, $b_k = \sqrt{k} \cot \sqrt{k}t = \dfrac{s'(t)}{s(t)}$ satisfying the Riccati equation

$$b_k' + b_k^2 + k = 0. \tag{2.11}$$

Now, putting $b = \dfrac{y'}{y}$, the equation (2.9) induces the Jacobi equation

$$y'' + \left(\frac{Ric + tr(\sigma^2)}{n-1}\right) y = 0.$$

Since σ extends to $t = 0$, by the standard ODE existence theorem, there exists a smooth solution $g(t)$ of this Jacobi equation with $g(0) = 0$, $g'(0) = 1$. The following lemma is given by Lemma 5 in [7].

LEMMA 2.1. *Let g and s be smooth functions satifying $g'' + kg \leq 0$ and $s'' + ks = 0$ with $g(0) = s(0)$, $g'(0) = s'(0)$ and g and s are positive for interval $(0, a)$. Suppose t_g and t_s are the first positive zeros of g and s. Then*

(i) $t_g \leq t_s$,
(ii) $g \leq s$ *on* $[0, t_s]$, *and equality at* t_0 *implies equality on* $[0, t_0]$,
(iii) $g'/g \leq s'/s$ *on* $(0, \min\{t_g, t_s\})$, *and equality at* t_0 *implies equality on* $(0, t_0]$.

3. Volume comparison theorem on space-times

Let M^n be the space-time given in Section 2 and let $p \in M$. For any $v \in E$, γ_v is also the radial geodesic with $\gamma_v(0) = p$ and $\gamma_v'(0) = v$. Choose a conjugate Jacobi tensor A along γ_v with $A(0) = 0$, $A'(0) = Id$. Putting $a(t) = \det A(t)$, since $tr(B) = tr(A'A^{-1}) = (\det A)'/\det A$, we have $a'(t) = tr(B)a(t)$. Setting $j_v(t) = a(t)^{\frac{1}{n-1}}$,

$$j_v'(t) = b(t)j_v(t) \tag{3.1}$$

where $b = \dfrac{1}{n-1} tr(B)$.

Moreover, let Q_k be an n-dimensional ($n \geq 2$) space-time of constant curvature $k > 0$. Then, for a conjugate Jacobi tensor A with $A(0) = 0$, $A'(0) = Id$, along any geodesic,

$$A(t) = \frac{1}{\sqrt{k}} \sin \sqrt{k} t Id. \tag{3.2}$$

Hence, $\det A(t) = (\dfrac{1}{\sqrt{k}} \sin \sqrt{k} t)^{n-1}$. Thus,

$$j(t) = (\det A(t))^{\frac{1}{n-1}} = \frac{1}{\sqrt{k}} \sin \sqrt{k} t > 0$$

for $0 < t < \dfrac{\pi}{\sqrt{k}}$. Therefore, we also have

$$j'(t) = b_k(t)j(t) \tag{3.3}$$

where $b_k(t) = \frac{1}{\sqrt{k}} \cot \sqrt{k}t$.

Let $\{v, e_1, e_2, ...e_{n-1}\}$ be an orthonormal basis of T_pM with $e_1, e_2, ..., e_{n-1}$ spacelike and extend them to the parallel frame fields P_i along γ_v with $P_i(0) = e_i$, $i = 1, 2, 3...., n-1$. Then $J_i(t) = (AP_i)(t)$ are linearly independent Jacobi fields along γ_v for $0 \leq t < cut_v(p)$. Since $J_i(t) = \alpha_{*|(t,0)}(\frac{d}{ds}) = D_{tv} \exp_p(te_i)$ for the normal variation α of γ_v,

$$\begin{aligned} t^{n-1} |\det \alpha_{*|(t,0)}| &= \| J_1(t) \wedge J_2(t) \wedge ......J_{n-1}(t) \| \\ &= \| AP_1 \wedge AP_2 \wedge .... \wedge AP_{n-1} \| \\ &= |\det A|. \end{aligned}$$

Thus, we have

$$|\det \alpha_{*|(t,0)}| = t^{-n+1} |\det A|. \tag{3.4}$$

Now, note that $j_v(0) = 0$, $j'_v(0) = 1$. For, $A(0) = 0$ implies $\det A(0) = 0$. Moreover, since $\lim_{t\to 0} B(t) = \lim_{t \to 0} \frac{1}{t} Id$, we also have

$$\lim_{t\to 0} j'_v(t) = \lim_{t \to 0} \frac{tr(B)}{n-1} j_v(t) = \lim_{t\to 0} \frac{1}{t} (\det A)^{\frac{1}{n-1}} = \lim_{t \to 0} |\det \alpha_{*|(t,0)}|^{\frac{1}{n-1}}.$$

Let $\overline{K}$ be a compact set in t_0E. Then $K = \exp_p t_0\overline{K}$ is also compact in $f^{-1}(t_0)$. Set $B_p^K(t_0) = \bigcup_{0 \leq t \leq t_0} \exp_p t\overline{K}$, and $V_p^K(t_0) = Vol(B_p^K(t_0))$.

Let du, dv be the volume elements of $Fut(T_pM)$, E respectively. Then we have the change of variables on space-times as follows;

LEMMA 3.1. $du = t^{n-1} dt dv$.

Proof. Define a differentiable map $\mu : \mathbf{R}^+ \times E \to Fut(T_pM)$ by $\mu(t, v) = tv$. Choose bases $\{\frac{\partial}{\partial t}, Y_1, Y_2, ..., Y_{n-1}\}$, $\{v, Y_1, Y_2, ..., Y_{n-1}\}$ for $T_{(t,v)}(\mathbf{R}^+ \times E)$, $T_{tv}(T_pM)$. Given $v \in E$, let $\{Y_1, Y_2, ..., Y_{n-1}\}$ be unit spacelike vectors which are orthogonal to v and we identify them on T_vE.

Note that $< v + sY_j, v + sY_j >= -1 + s^2 < 0$ for sufficiently small s so that $v + sY_j$ will be a future timelike vector for the s.

To calculate $\det(\mu_*)$ where $\mu_* : T_{(t,v)}(\mathbf{R} \times E) \to T_{tv}(T_pM)$, we start with $c(s) = (t + s, v)$ to obtain $\mu_* \frac{\partial}{\partial t} = (\mu \circ c)'(0) = \frac{d}{ds}(tv + sv)_{|s=0} = v$.

To calculate $\mu_* Y_j$, we form the curve

$$c(s) = (t, \frac{v + sY_j}{\| v + sY_j \|}) = (t, \frac{v}{\sqrt{1 - s^2}} + \frac{s}{\sqrt{1 - s^2}} Y_j)$$

where

$$\begin{aligned} \| v + sY_j \| &= \sqrt{- < v + sY_j, v + sY_j >} \\ &= \sqrt{- < v, v > -2s < v, Y_j > -s^2 < Y_j, Y_j >} = \sqrt{1 - s^2}. \end{aligned}$$

Hence, $\mu \circ c(s) = \frac{1}{\sqrt{1 - s^2}} tv + \frac{s}{\sqrt{1 - s^2}} tY_j$.

Thus, we obtain

$$\mu_* \frac{\partial}{\partial Y_j} = (\mu \circ c)'(0) = tY_j.$$

Moreover, we may have the matrix of μ_*

$$\begin{pmatrix} 1 & 0 & 0 \ldots\ldots & 0 \\ 0 & t & 0 \ldots\ldots & 0 \\ . & . & \ldots\ldots\ldots & . \\ 0 & 0 & 0 \ldots\ldots & t \end{pmatrix}$$

and $\det(\mu_*) = t^{n-1}$. □

Now, we define a differential map $F : M \to Q_k$ by $F = \exp_p \circ i \circ \exp_p^{-1}$ where $i : T_pM \to T_{F(p)}Q_k$ is a linear isometry to show the Bishop type volume comparison theorem on Lorentzian manifold.

THEOREM 3.2. *Let M^n be a globally hyperbolic space-time with its Ricci curvature $\geq (n-1)k > 0$ for some $k > 0$. Then, if $V_{F(p)}(r_0) = Vol(B_{F(p)}^{F(K)}(r_0))$ for $0 \leq r_0 \leq \min\{Cut_\nu(p) \mid \nu \in \bar{K}\}$,*

$$V_p^K(r_0) \leq V_{F(p)}(r_0) \tag{3.5}$$

and equality implies that for any $0 \leq t \leq r_0$, $B_p^K(t)$ *is isometric to* $B_{F(p)}^{F(K)}(t)$.

Proof. From Lemma 3.1,

$$V_p^K(r_0) = \int_{B_0^{\overline{K}}} |\det \alpha_{*|(t,0)}| du = \int_{\overline{K}} \int_0^{r_0} |\det \alpha_{*|(t,0)}| t^{n-1} dt dv$$
$$= \int_{\overline{K}} \int_0^{r_0} |\det A| dt dv = \int_{\overline{K}} \int_0^{r_0} j_v(t)^{n-1} dt dv.$$

and $V_{F(p)}(r_0) = \int_{i(\overline{K})} \int_0^{r_0} (j(t))^{n-1} dt dv.$

Note that $j(0) = 0$, $j'(0) = 1$ and $j_v(0) = 0$, $j_v'(0) = 1$. By Lemma 2.1 (ii), the inequality holds. Now suppose $V_p^K(r_0) = V_{F(p)}(r_0)$. Then $\int_0^{r_0} j_v(t)dt = \int_0^{r_0} j(t)dt$.

Since $j_v(t)$ and $j(t)$ are positive in $(0, \min\{t_g, t_s\})$ where t_g and t_s are given in Lemma 2.1, we have $j_v = j = \frac{1}{\sqrt{k}} \sin \sqrt{k}t$ on $[0, r_0]$ for any fixed $v \in E$.

Hence, $b = b_k$ and $\frac{Ric}{n-1} + \frac{tr(\sigma^2)}{n-1} = k$ from the equations (2.9) and (2.11).

Since $Ric \geq (n-1)k$,

$$\frac{Ric}{n-1} + \frac{tr(\sigma^2)}{n-1} \geq \frac{Ric}{n-1} \geq k,$$

which implies $tr(\sigma^2) = 0$, i.e., $\sigma = 0$ since σ is self-adjoint.

Moreover, $B(t) = bId = b_k Id$.

From (2.2), we have $R = kId$. From (3.2), $J_i(t) = \frac{1}{\sqrt{k}} \sin \sqrt{k} t P_i(t)$ along γ_v where $P_i(t)$ are parallel fields of the spacelike orthonormal vectors e_i along γ_v. Therefore, $F = \exp_p \circ i \circ \exp_p^{-1} : B_p^K(t) \to B_{F(p)}^{F(K)}(t)$ is an isometry on $[0, r_0]$. □

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