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A NOTE ON A GENERAL MAXIMAL OPERATOR

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1. Introduction

Let μ be a positive Borel measure on $\mathbb{R}^n$ which is positive on cubes. For any cube $Q \subset \mathbb{R}^n$, a Borel measurable nonnegative function φ_Q, supported and positive a.e. with respect to μ in Q, is given. We consider a maximal function

$$M_\mu f(x) = \sup \int \varphi_Q |f| d\mu$$

where the supremum is taken over all φ_Q such that $x \in Q$.

This operator was studied in [6], [4], [5] in connetion with the Muckenhaupt's Ap-condition [7], fractional maximal operator and sph- erical maximal function.

In this note we study some more properties of M_μ and some special cases.

Throughout this paper Q will denote a cube in $\mathbb{R}^n$ with sides parallel to coordinate axes.

2. A condition related to the two-weight strong-type (p, q) inequality

In this section we first give a necessary condition for the two-weight strong-type (p, q) inequality for M_μ when $p > 1$ and then we show that it is also a sufficient condition for the two-weight strong-type (p, ∞)

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inequality for M_μ, restricted to dyadic cubes. The condition is a modification of Sawyer's condition [8].

Throughout this section w and ν are positive Borel measure on $\mathbb{R}^n$, positive on cubes.

PROPOSITION 1. *If* $\|M_\mu f\|_{L^q(w)} \le C\|f\|_{L^p(\nu)}$ *for* $p > 1$ *and* $q \ge 1$, *then* $\mu \ll \nu$ *and*

$$\left\|M_\mu\left(\varphi_Q^{p'-1}\left(\frac{d\mu}{d\nu}\right)^{p'-1}\right)\right\|_{L^q(w)} \le C\left\|\varphi_Q^{p'-1}\left(\frac{d\mu}{d\nu}\right)^{p'-1}\right\|_{L^p(\nu)} < \infty$$

for all φ_Q, *where* p' *is the conjugate exponent of* p.

Proof. First suppose it is not true that $\mu \ll \nu$. Then there exists a Borel set E such that $\nu(E) = 0$ but $\mu(E) > 0$. Let $f = \chi_E$. Then $\|f\|_{L^p(\nu)} = 0$ but $M_\mu f(x) > 0$ for all $x \in \mathbb{R}^n$. Therefore, $\|M_\mu f\|_{L^q(w)} > 0$ unless $w = 0$. So, we must have $\mu \ll \nu$.

Now suppose $\|\varphi_Q^{p'-1}(\frac{d\mu}{d\nu})^{p'-1}\|_{L^p(\nu)} = \infty$ for some φ_Q. This means $\int \varphi_Q^{p'}(\frac{d\mu}{d\nu})^{p'} d\nu = \infty$. So, there exists $f_n \in L^p(\nu)$ such that $\|f_n\|_{L^p(\nu)} = 1$ and $\int f_n\varphi_Q\frac{d\mu}{d\nu}d\nu = \int f_n\varphi_Q d\mu \to \infty$ as $n \to \infty$. Since $M_\mu f_n(x) \ge \int f_n\varphi_Q d\mu$ for every $x \in Q$, $\|M_\mu f_n\|_{L^q(w)} \to \infty$ as $n \to \infty$. Since $\|f_n\|_{L^p(\nu)} = 1$ for every n, this shows

$$\left\|\varphi_Q^{p'-1}\left(\frac{d\mu}{d\nu}\right)^{p'-1}\right\|_{L^p(\nu)} < \infty \quad \text{for all} \quad \varphi_Q.$$

The inequality

$$\left\|M_\mu\left(\varphi_Q^{p'-1}\left(\frac{d\mu}{d\nu}\right)^{p'-1}\right)\right\|_{L^q(w)} \le C\left\|\varphi_Q^{P'-1}\left(\frac{d\mu}{d\nu}\right)^{p'-1}\right\|_{L^p(\nu)}$$

is obvious if we put $f = \varphi_Q^{p'-1}(\frac{d\mu}{d\nu})^{p'-1}$ in the hypothesis. □

Now we write $M_{d,\mu}f$ for $M_\mu f$, restricted to dyadic cubes, that is,

$$M_{d,\mu}f(x) = \sup \int \varphi_Q|f|d\mu,$$

where the sup is taken over all φ_Q such that $x \in Q$ and Q is dyadic.

In the following, we restrict ourselves to the case when $\nu \ll \mu$ and to avoid the trivial special cases arising and for the simplicity, we assume that $\varphi_Q > 0$ a.e. on Q with respet to ν.

Throughtout this paper, p' denotes the conjugate exponent of p.

PROPOSITION 2. *Suppose $\mu \ll \nu$ and for $p > 1$,*

$$\left\| M_{d,\mu}\left(\varphi_Q^{p'-1}\left(\frac{d\mu}{d\nu}\right)^{p'-1}\right)\right\|_{L^\infty(w)} \le C \left\|\varphi_Q^{p'-1}\left(\frac{d\mu}{d\nu}\right)^{p'-1}\right\|_{L^p(\nu)} < \infty$$

for all Q. Then $\|M_{d,\mu} f\|_{L^\infty(w)} \le C\|f\|_{L^p(\nu)}$.

Proof. Let $f \in L^p(\nu)$ and fix $\lambda > 0$. Consider $\Omega = \{M_{d,\mu}^R f > \lambda\}$, where $M_{d,\mu}^R f(x)$ is the $M_{d,\mu} f(x)$ restricted to the dyadic cubes with side length $\le R$. If $M_{d,\mu}^R f(x) > \lambda$, then there exists a dyadic cube Q_x containing x such that side length of $Q_x \le R$ and $\int \varphi_{Q_x}|f|d\mu > \lambda$. Then we have

$$\Omega = \cup_{x\in\Omega}\, Q_x.$$

Let $D = \{Q_x \mid x \in \Omega\}$. Then every cube in D is contained in some maximal cube in D and the maximal cubes are mutually nonoverlapping. Therefore, $\Omega = \cup\, Q_k$, where the Q_k's are maximal cubes in D and so $\mathring{Q}_k \cap \mathring{Q}_j = \emptyset$ ($\mathring{Q}_k$ denotes the interior of Q_k) if $k \ne j$ and $\int \varphi_{Q_k}|f|d\mu > \lambda$.

$$\begin{aligned}
\lambda &< \int \varphi_{Q_k}|f|d\mu = \int_{Q_k} |f|\varphi_{Q_k}\frac{d\mu}{d\nu}d\nu \\
&\le \left(\int_{Q_k} |f|^p d\nu\right)^{\frac{1}{p}} \left(\int_{Q_k} \varphi_{Q_k}^{p'}\left(\frac{d\mu}{d\nu}\right)^{p'} d\nu\right)^{\frac{1}{p'}} \quad (1)\\
&\qquad \text{by Hölder's inequality} \\
&= \left(\int_{Q_k} |f|^p d\nu\right)^{\frac{1}{p}} \left\|\varphi_{Q_k}{}^{p'-1}\left(\frac{d\mu}{d\nu}\right)^{p'-1}\right\|_{L^p(\nu)}^{\frac{p}{p'}} < \infty
\end{aligned}$$

from the hypothesis. For every k,

$$\begin{aligned}
M_\mu\left(\varphi_{Q_k}{}^{p'-1}\left(\frac{d\mu}{d\nu}\right)^{p'-1}\right) &\le \int \varphi_{Q_k}\varphi_{Q_k}{}^{p'-1}\left(\frac{d\mu}{d\nu}\right)^{p'-1} d\mu \;\text{ on }\; Q_k \\
&= \int \varphi_{Q_k}{}^{p'}\left(\frac{d\mu}{d\nu}\right)^{p'} d\nu \;\text{ since }\; \nu \ll \mu \\
&= \left\|\varphi_{Q_k}{}^{p'-1}\left(\frac{d\mu}{d\nu}\right)^{p'-1}\right\|_{L^p(\nu)}^p \;\text{ on }\; Q_k.
\end{aligned}$$

So, since $w(Q_k) > 0$,

$$\left\| M_\mu \left({\varphi_{Q_k}}^{p'-1} \left(\frac{d\mu}{d\nu}\right)^{p'-1} \right) \right\|_{L^\infty(w)} \geq \left\| {\varphi_{Q_k}}^{p'-1} \left(\frac{d\mu}{d\nu}\right)^{p'-1} \right\|^p_{L^p(\nu)}.$$

Since

$$\left\| M_\mu \left({\varphi_{Q_k}}^{p'-1} \left(\frac{d\mu}{d\nu}\right)^{p'-1} \right) \right\|_{L^\infty(w)} \leq C \left\| {\varphi_{Q_k}}^{p'-1} \left(\frac{d\mu}{d\nu}\right)^{p'-1} \right\|_{L^p(\nu)} < \infty,$$

$$\left\| {\varphi_{Q_k}}^{p'-1} \left(\frac{d\mu}{d\nu}\right)^{p'-1} \right\|^p_{L^p(\nu)} \leq C \left\| {\varphi_{Q_k}}^{p'-1} (\frac{d\mu}{d\nu})^{p'-1} \right\|_{L^p(\nu)}$$

for every Q_k.

Therefore, since $\|{\varphi_{Q_k}}^{p'-1}(\frac{d\mu}{d\nu})^{p'-1}\|_{L^p(\nu)} \neq 0$,

$$\left\| {\varphi_{Q_k}}^{p'-1} \left(\frac{d\mu}{d\nu}\right)^{p'-1} \right\|^{p-1}_{L^p(\nu)} \leq C \quad \text{when} \quad p > 1$$

$$\leq \frac{C}{\lambda} \|f\|_{L^p(\nu)} \left\| {\varphi_{Q_k}}^{p'-1} \left(\frac{d\mu}{d\nu}\right)^{p'-1} \right\|^{\frac{p}{p'}}_{L^p(\nu)}$$

by (1). Since $0 < \|{\varphi_{Q_k}}^{p'-1}(\frac{d\mu}{d\nu})^{p'-1}\|_{L^p(\nu)} < \infty$, we have $1 \leq \frac{C}{\lambda}\|f\|_{L^p(\nu)}$, i.e., $\lambda \leq C\|f\|_{L^p(\nu)}$.

Since $\Omega = \cup Q_k$, this implies $\|M^R_{d,\mu} f\|_{L^\infty(w)} \leq C\|f\|_{L^p(\nu)}$. R is arbitrary. Therefore, we have $\|M_{d,\mu} f\|_{L^\infty(w)} \leq C\|f\|_{L^p(\nu)}$. □

For any cube Q, let Q^d denote the smallest dyadic cube containing Q. Suppose there exist positive constants C_1 and C_2, depending only on the measures s.t.

$$(2) \qquad C_1 \varphi_{Q^d} \leq \varphi_Q \leq C_2 \varphi_{Q^d} \quad \text{on} \quad Q.$$

Then for any cube Q containing x

$$\int \varphi_Q |f| d\mu \leq C_2 \int \varphi_{Q^d} |f| d\mu \leq C_2 M_{d,\mu} f(x)$$

Therefore,

$$M_\mu f(x) \leq C_2 M_{d,\mu} f(x)$$

Thus we have

PROPOSITION 3. *Suppose* (2) *holds and assume* $\mu \ll \nu$ *and for* $p > 1$ *and for all* φ_Q

$$(3) \quad \left\| M_\mu \left(\varphi_Q^{p'-1} \left(\frac{d\mu}{d\nu} \right)^{p'-1} \right) \right\|_{L^\infty(w)} \leq C \left\| \varphi_Q^{p'-1} \left(\frac{d\mu}{d\nu} \right)^{p'-1} \right\|_{L^p(\nu)} < \infty.$$

Then $\|M_\mu f\|_{L^\infty(w)} \leq C \|f\|_{L^p(\nu)}$.

EXAMPLE. Let $\varphi_Q(x) = \mu(Q)^{\frac{\alpha}{n}-1}\chi_Q$, where $0 \leq \alpha < n$. Then $M_\mu f(x) = \sup_{x \in Q} \mu(Q)^{\frac{\alpha}{n}-1} \int_Q |f| d\mu$ is the weighted fractional maximal operator. If μ satisfies the doubling condition, then for every cube Q

$$\varphi_{Q^d} \leq \varphi_Q \leq C_{\mu,n} \varphi_{Q^d} \quad \text{on} \quad Q,$$

where $C_{\mu,n}$ is à constant depending only on μ and the dimension n. Therefore, in this case Proposition 3 holds and (3) reduces to the Sawyer's condition [8]. So we will put the Sawyer's theorem as a corollary.

COROLLARY. [8] *Suppose* μ *satisfies the doubling condition and* $p > 1$. *If* $0 \leq \alpha < n$, *define* $M_{\mu,\alpha} f(x) = \sup_{x \in Q} \mu(Q)^{\frac{\alpha}{n}-1} \int_Q |f| d\mu$.

Then $\|M_{\mu,\alpha} f(x)\|_{L^\infty(w)} \leq C \|f\|_{L^p(\nu)}$ *for all* $f \in L^p(\nu)$ *if and only if* $\mu \ll \nu$ *and* $\|\chi_Q M_{\mu,\alpha}(\chi_Q (\frac{d\mu}{d\nu})^{p'-1})\|_{L^\infty(w)} \leq C \|\chi_Q (\frac{d\mu}{d\nu})^{p'-1}\|_{L^p(\nu)} < \infty$ *for all cubes* $Q \subset \mathbb{R}^n$.

3. $L^{p,q}$ norm inequality for the Hardy-Littlewood maximal operator

In this section we consider the special case when w and ν are equal weights and φ_Q is specifically given.

Let $d\mu = \mathbf{u}(x)dx$ where $\mathbf{u}(x)$ is a function s.t. $0 < \mathbf{u} < \infty$ a.e. with respect to the Lebesgue measure on $\mathbb{R}^n$.

We'll first give some definitions in [2].

DEFINITION 1. The nonincreasing rearrangement $g^*_\mu(t)$ of a function g with respect to the measure μ is defined as

$$g^*_\mu(t) = \inf \{ s \ : \ \mu(\{x \ : \ |g(x)| > s\}) \leq t \}$$

DEFINITION 2. $L^{p,q}$ is the collection of all functions g with $\|g\|_{p,q;\mu} < \infty$, where

$$\|g\|_{p,q} = \|g\|_{p,q;\mu} = \begin{cases} \left(\frac{q}{p}\int_0^\infty (t^{\frac{1}{p}} g_\mu^*(t))^q \frac{dt}{t}\right)^{\frac{1}{q}}, & 1 \le p < \infty,\ 1 \le q < \infty \\ \sup_{t>0} t^{\frac{1}{p}} g_\mu^*(t), & 1 \le p < \infty,\ q = \infty. \end{cases}$$

If $\varphi_Q(x) = \frac{1}{|Q|}\frac{\chi_Q(x)}{\mathbf{u}(x)}$, then $M_\mu f(x)$ becomes the ordinary Hardy-Littlewood maximal function $Mf(x)$ of f. Here $|Q|$ denotes the Lebesgue measure of Q.

Let $dw = w(x)dx$ and $\Phi(t) = \sup_Q\{w(Q)\varphi_{Q,\mu}^*(w(Q)t)\}$.

Then we have $\Phi \in L^{p',1}$ implies that $\|Mf\|_{L^p(w)} \le C\|f\|_{L^p(\mu)}$. (For the proof, we refer to [6].

We now consider this Hardy-Littlewood maximal operator Mf for the single weight problem, i.e., when $w = \mu$. Throughout this section, the norms are all with respect to the measure $d\mu = \mathbf{u}(x)dx$.

DEFINITION 3. [7] We say $\mathbf{u} \in A_p$ if

$$\left(\int_Q \mathbf{u}(x)dx\right)\left(\int_Q \mathbf{u}(x)^{-\frac{1}{p-1}}dx\right)^{p-1} \le C|Q|^p \qquad \text{if } 1 < p < \infty$$

$$\int_Q \mathbf{u}(x)dx \le C|Q| \ \operatorname*{ess\,inf}_{x\in I} \mathbf{u}(x) \qquad \text{if } p = 1,$$

for any cube Q, where C is a constant independent of Q.

DEFINITION 4. [1] Suppose either $1 < p < \infty$ and $1 \le q \le \infty$ or $p = q = 1$. A nonnegative, locally integrable function $\mathbf{u}(x)$ is in $A(p,q)$ if there exists a constant C such that for any cube Q,

$$\|\chi_Q\|_{p,q}\ \|\chi_Q \mathbf{u}^{-1}\|_{p',q'} \le C|Q|.$$

We note that $\mathbf{u} \in A(p,p)$ if and only if $\mathbf{u} \in A_p$.

We now list some theorems in [1] as lemmas ;

LEMMA 1. [1] *If either* $1 < p < \infty$ *and* $1 \le q \le \infty$ *or* $p = q = 1$, *then* $\|Mf\|_{p,\infty} \le C\|f\|_{p,q}$ *implies* $\mathbf{u} \in A(p,q)$.

LEMMA 2. [1] *If* $1 \le q \le p < \infty$, *then* $\mathbf{u} \in A(p,q)$ *implies* $\|Mf\|_{p,\infty} \le C\|f\|_{p,q}$.

LEMMA 3. [1] *If* $1 < p < \infty$ *and* $1 < q \le \infty$, *then* $\mathbf{u} \in A(p,q)$ *implies* $\|Mf\|_{p,s} \le C\|f\|_{p,s}$ *for* $1 \le s \le \infty$.

LEMMA 4. [1] *If either* $1 < p < \infty$ *and* $1 \le q \le \infty$ *or* $p = q = 1$, *then* $\mathbf{u} \in A(p,q)$ *if and only if* $\|Mf\|_{p,\infty} \le C\|f\|_{p,q}$.

Using the above Lemmas we are able to see the following propositions.

PROPOSITION 4. *If either* $1 < p < \infty$ *and* $1 \le q \le \infty$ *or* $p = q = 1$, *then* $\|Mf\|_{p,q} \le C\|f\|p,q$ *implies* $\mathbf{u} \in A(p,q)$.

Proof. $\|Mf\|_{p,\infty} \le \|Mf\|_{p,q}$ for all $1 \le p,\ q \le \infty$.
So from Lemma 1 it holds. □

From Lemmas 1 & 3 and from the fact $\|Mf\|_{p,\infty} \le \|Mf\|_{p,q}$ for every $1 \le q \le \infty$, we have,

PROPOSITION 5. *For* $1 < p < \infty$ *and* $1 < q \le \infty$, *we have* $\|Mf\|_{p,q} \le C\|f\|_{p,q}$ *if and only if* $\mathbf{u} \in A(p,q)$.

In [3] we have the following as a theorem.
" **For** $1 \le p \le q < \infty$ **and** $1 \le r \le \infty$, **if** μ **is a doubling measure, then** $\|Mf\|_{q,\infty;\mu} \le B\|f\|_{p,r;\nu}$ **if and only if** $\Phi \in L^{p',r'}(0,\infty)$"
From this fact, we know that for any $1 \le p < \infty$ and $1 \le q \le \infty$, if μ is a doubling measure, then $\|Mf\|_{p,\infty;\mu} \le B\|f\|_{p,q;\nu}$ if and only if $\Phi \in L^{p',q'}(0,\infty)$.
Therefore from Lemma 4, we can see the following relationship between $A(p,q)$ condition and Φ.

PROPOSITION 6. *If either* $1 < p < \infty$ *and* $1 \le q \le \infty$ *or* $p = q = 1$ *and if* μ *is a doubling measure, then* $\mathbf{u} \in A(p,q)$ *if and only if* $\Phi \in L^{p',q'}(0,\infty)$.

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