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CONTINUITY OF THE OPERATORS ON THE SPACES $\mathcal{D}_p$ AND AND THE DUAL SPACES $\mathcal{D}'_p$

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0. Introduction

First we recall some properties of the spaces $\mathcal{D}_p$ and study the continuity of the operators on $\mathcal{D}_p$. We also consider the continuity of the operations on the dual spaces $\mathcal{D}'_p$ with the weak topology.

1. Definitions and notations

The *normalized Lebesgue measure* on R^n is the measure m_n defined by $dm_n(x) = (2\pi)^{-n/2}dx$. The usual Lebesgue spaces L^p, or $L^p(R^n)$, will be normed by means of m_n:

$$||f||_{L^p} = \left\{ \int_{R^n} |f|^p dm_n \right\}^{1/p} \qquad (1 \leq p < \infty).$$

For each $t \in R^n$, the *character* e_t is the function defined by

$$e_t(x) = e^{itx} = \exp\{i(t_1x_1 + \cdots + t_nx_n)\} \qquad (x \in R^n).$$

The *Fourier transform* of the function $f \in L^1(R^n)$ is the function $\hat{f}$ defined by

$$\hat{f}(t) = \int_{R^n} f e_{-t} dm_n \qquad (t \in R^n).$$

If α is a multi-index, then

$$D_\alpha = (i)^{-|\alpha|}D^\alpha = \left(\frac{1}{i}\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{1}{i}\frac{\partial}{\partial x_n}\right)^{\alpha_n}.$$

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If P is a polynomial of n variables, with complex coefficients, say

$$P(\xi) = \sum C_\alpha \xi^\alpha = \sum C_\alpha \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n},$$

the differential operators $P(D)$ and $P(-D)$ are defined by

$$P(D) = \sum C_\alpha D_\alpha, \qquad P(-D) = \sum (-1)^{|\alpha|} C_\alpha D_\alpha.$$

The relation $S_1 \Subset S_2$ shall mean that the closure of S_1 is compact and contained in the interior of S_2. If $\{S_j\}_{j=1}^{\infty}$ is a sequence of sets, the relation $S_j \nearrow\!\!\nearrow S$ shall mean that $S_j \Subset S_{j+1} (j = 1, 2, \cdots)$ and that $S = \cup S_j$. Let p be a real-valued function on R^n, continuous at the origin and having the property

$$(\alpha) \qquad 0 = p(0) = \lim_{x \to 0} p(x) \leq p(\xi + \eta) \leq p(\xi) + p(\eta) \quad (^\forall \xi, \eta \in R^n).$$

DEFINITION 1.1. Let $\mathcal{M}_0 = \mathcal{M}_0(n)$ be the set of all continuous real-valued functions p on R^n satisfying the conditions (α) and

$$(\beta) \qquad J_n(p) = \int_{|\xi| \geq 1} \frac{p(\xi)}{|\xi|^{n+1}} d\xi < \infty.$$

DEFINITION 1.2. Let p satisfy (α). If $\phi \in L^1(R^n)$ and if λ is a real number, we write

$$||\phi||_\lambda = ||\phi||_\lambda^{(p)} = \int |\hat{\phi}(\xi)| e^{\lambda p(\xi)} d\xi.$$

Let $\mathcal{D}_p$ be the set of all ϕ in $L^1(R^n)$ such that ϕ has compact support and $||\phi||_\lambda < \infty$ for all $\lambda > 0$. The elements of $\mathcal{D}_p$ will be called test functions.

DEFINITION 1.3. Let p_1 and p_2 be the elements in $\mathcal{M}_0(n)$. If for some real a and positive b we have $p_2(\xi) \leq a + bp_1(\xi) \quad (^\forall \xi \in R^n)$. Then p_2 is said to be *dominated* by p_1 with some constant translation. We denote this by $p_2 \prec p_1$.

DEFINITION 1.4. If K is a compact subset of R^n, $\mathcal{D}_p(K) = \{\phi \in \mathcal{D}_p;\ \text{supp}\ \phi \subset K\}$. Note that the space $\mathcal{D}_p(K)$ is a Fréchet space under the natural linear structure and the seminorms $\|\cdot\|_m \quad (m = 1, 2, \cdots)$.

DEFINITION 1.5. If Ω is an open subset of R^n and if $K_\nu \nearrow\nearrow \Omega$ we define $\mathcal{D}_p(\Omega)$ as the inductive limit of the Fréchet spaces $\mathcal{D}_p(K_\nu)$, i.e., $\mathcal{D}_p(\Omega) = \text{ind}\lim_{K_\nu \Subset \Omega} \mathcal{D}_p(K_\nu)$.

DEFINITION 1.6. Let $\mathcal{M} = \{p \in \mathcal{M}_0(n) : p \text{ satisfy condition } (\gamma)\}$:

$$(\gamma) \qquad p_0 \prec p, \quad \text{where} \quad p_0(x) = \ln(1 + |x|) \quad (x \in R^n).$$

2. The spaces $\mathcal{D}_p$ and continuity

PROPOSITION 2.1. *$p_2 \prec p_1$ if and only if there are some real a and positive b such that $a + bp_2(x) \leq p_1(x)$ for all $x \in R^n$.*

Proof. It is obvious

THEOREM 2.2. ([2] Thm.1.3.18) *If $p_2 \prec p_1$, then $\mathcal{D}_{p_1} \subset \mathcal{D}_{p_2}$ and $\mathcal{D}_{p_1}(\Omega)$ is dense in $\mathcal{D}_{p_2}(\Omega)$ for each open $\Omega \subset R^n$. Conversely, if for some compact $K \subset R^n$ with $\mathring{K} \neq \phi, \mathcal{D}_{p_1}(K) \subset \mathcal{D}_{p_2}(K)$, then $p_2 \prec p_1$.*

Proof. For some real a and positive b, we have $p_2(x) \leq a + bp_1(x)(x \in R^n)$ and hence

$$\|f\|_\lambda^{(p_2)} \leq e^{\lambda a}\|f\|_{\lambda b}^{(p_1)} < \infty \qquad (f \in \mathcal{D}_{p_1}).$$

Hence $\mathcal{D}_{p_1} \subset \mathcal{D}_{p_2}$. Let $u \in \mathcal{D}_{p_2}(\Omega)$ and let $u_\epsilon = u * f_\epsilon$ with $f \in \mathcal{D}_{p_1'}(\Omega)$, where $f_\epsilon(x) = \epsilon^{-n} f(x/\epsilon)$ and $p_1'(x) = \sup_{|\xi| \leq |x|} p_1(\xi)$. Then $u_\epsilon \in \mathcal{D}_{p_1}(\Omega)$ and $\lim_{\epsilon \to 0} \|u - u_\epsilon\|_\lambda^{(p_2)} = 0$.

To prove the converse, choose $K \subset R^n$ compact with $\mathring{K} \neq \phi$ such that $\mathcal{D}_{p_1}(K) \subset \mathcal{D}_{p_2}(K)$. The inclusion map of $\mathcal{D}_{p_1}(K)$ into $\mathcal{D}_{p_2}(K)$ is closed and hence continuous by the closed graph theorem. Therefore, for some positive constants b and b' we have

$$b'\|f\|_1^{(p_2)} \leq \|f\|_b^{(p_1)} \qquad (f \in \mathcal{D}_{p_1}(K)) \qquad \cdots (*_1).$$

Let $t_0 \in R^n$ and g be a nontrivial element in $\mathcal{D}_{p_1}(K)$ and define $f = ge_{t_0}$. Then $\hat{f}(t) = \hat{g}(t - t_0)$. We get

$$||f||_1^{(p_1)} \leq e^{bp_1(t_0)}||g||_b^{(p_1)} \qquad \cdots(*_2)$$

and

$$||f||_b^{(p_2)} \geq e^{p_2(t_0)} \int |\hat{g}(t)|e^{-p_2(-t)}dt \geq e^{p_2(t_0)}\delta||g||_1^{(p_2)} \qquad \cdots(*_3)$$

for some $\delta > 0$. Hence, from $(*_1),(*_2)$ and $(*_3)$, we have

$$e^{p_2(t_0)}\delta||g||_1^{(p_2)} \leq \frac{1}{b'}e^{bp_1(t_0)}||g||_b^{(p_1)} \qquad \text{and hence}$$

$$p_2(t_0) + \ln \delta||g||_1^{(p_2)} \leq bp_1(t_0) + \ln \frac{||g||_b^{(p_1)}}{b'}.$$

Hence we derive $p_2 \prec p_1$ with $a = \ln \frac{||g||_b^{(p_1)}}{b'} - \ln \delta||g||_1^{(p_2)}$.

COROLLARY 2.3. *Let $p \in \mathcal{M}_0(n)$. Then $\check{p} \in \mathcal{M}_0(n)$, where $\check{p}(x) = p(-x)$. $\check{p} \prec p$ if and only if $p \prec \check{p}$*

COROLLARY 2.4. *Let $p \in \mathcal{M}_0(n)$. Then $\mathcal{D}_p(\Omega) \subset \mathcal{D}(\Omega)$ for every open Ω in R^n (or for some non-trivial Ω) if and only if $p_0 \prec p$, where $p_0(x) = \ln(1 + |x|)$ $(x \in R^n)$.*

DEFINITION 2.5. ([2], Def 1.3.22) Let $\mathcal{M} = \{p \in \mathcal{M}_0(n); p$ satify condition $(\gamma)\}$:

$$(\gamma) \qquad p_0 \prec p \quad \text{where} \quad p_0(x) = \ln(1 + |x|) \quad (x \in R^n).$$

The translation operators τ_x are drfined by $(\tau_x)f(y) = f(y - x)$ $(x, y \in R^n)$.

THEOREM 2.6. *Let $p \in \mathcal{M}_0(n)$ and let $x \in R^n$ be given. Then the mapping T_x from $\mathcal{D}_p$ into $\mathcal{D}_p$ defined by $T_x(f) = \tau_x f$ $(f \in \mathcal{D}_p)$ is continuous and in fact an isometry.*

Proof. Since $(\tau_x f)\hat{} = e_{-x}\hat{f}$, we have

$$||T_x(f)||_\lambda^{(p)} = ||\tau_x f||_\lambda^{(p)} = \int |e_{-x}\hat{f}|e^{\lambda p(t)}dt = ||f||_\lambda^{(p)}.$$

THEOREM 2.7. *Let $p \in \mathcal{M}_0(n)$ and let $x \in R^n$ be given. Then the mapping F_x from $\mathcal{D}_p$ into $\mathcal{D}_p$ defined by $F_x(f) = e_x f \quad (f \in \mathcal{D}_p)$ is linear and continuous.*

Proof. Since $\widehat{(e_x f)} = \tau_x \hat{f}, \quad ||e_x f||_\lambda^{(p)} \le e^{\lambda p(x)} ||f||_\lambda^{(p)}$.

THEOREM 2.8. *Let $p \in \mathcal{M}$. If P is a polynomial and $g \in \mathcal{D}_p$, then each of the three mappings*

$$F_g : f \to fg, \quad F_{*g} : f \to f * g, \quad P(D) : f \to P(D)f$$

is a continuous linear mapping of $\mathcal{D}_p$ into $\mathcal{D}_p$.

Proof. Since $\widehat{(fg)} = \hat{f} * \hat{g}, ||fg||_\lambda^{(p)} \le ||f||_\lambda^{(p)} ||g||_\lambda^{(p)}$. Hence, F_g is continuous. Since $\widehat{(f * g)} = \hat{f}\hat{g}, ||f * g||_\lambda^{(p)} \le ||\hat{g}||_\infty ||f||_\lambda^{(p)} \le ||g||_{L^1} ||f||_\lambda^{(p)}$. Therefore, F_{*g} is continuous. Finally, since $\widehat{(P(D)f)} = P\hat{f}$, and since $|P(t)| \le C\exp Ip(\xi)$ for some $I > 0$, and constant C, we hawe

$$\begin{aligned} ||P(D)f||_\lambda^{(p)} &= \int |P(t)\hat{f}(t)| e^{\lambda p(t)} dt \\ &\le C \int |\hat{f}(t)| e^{(\lambda + I)p(t)} dt \\ &= C ||f||_{\lambda+I}^{(p)}. \end{aligned}$$

Hence, $P(D)$ is continuous.

3. The spaces $\mathcal{D}'_p$ and continuity

We recall that $\check{p}(\xi) = p(-\xi)$ and note that if $p \in \mathcal{M}$, then $\check{p} \in \mathcal{M}$. The dual space $\mathcal{D}'_p$ of the space $\mathcal{D}_p$ is given the weak topology, that is the topology given by the system of semi-norms $\{||\cdot||_\phi : ||u||_\phi = |u(\phi)|, \quad \phi \in \mathcal{D}_p\}$.

DEFINITION 3.1. $\mathcal{E}_p(\Omega)$ is the set of all complex-valued functions ϕ in Ω such that if $\psi \in \mathcal{D}_p(\Omega)$, then $\psi\phi \in \mathcal{D}_p(\Omega)$. The topology in $\mathcal{E}_p(\Omega)$ is given by the semi-norms $||\cdot||_{\lambda,\psi}$ defined by $||\phi||_{\lambda,\psi} = ||\psi\phi||_\lambda^{(p)} \qquad ({}^\forall \lambda > 0, {}^\forall \psi \in \mathcal{D}_p(\Omega))$.

DEFINITION 3.2. ([2], Def 1.6.8) $\mathcal{D}'_{p,F}(\Omega)$ is the set of all $u \in \mathcal{D}'_p(\Omega)$ with the property that $\lambda > 0$ can be chosen so that for each compact $K \subset \Omega$ there exists C such that $|u(\phi)| \le C||\phi||_\lambda^{(p)} \quad (^\forall\phi \in \mathcal{D}_p(K))$.

DEFINITION 3.3. ([2], Def 1.8.10) Let $p \in \mathcal{M}$. We denote by $\mathcal{F}_p$ the set of all elements $u \in \mathcal{D}'_p$ such that for some measurable function U with $\int |U(\xi)|e^{-\lambda p(\xi)}d\xi < \infty$ for some $\lambda > 0$ we have $u(\phi) = \int U(\xi)\hat{\phi}(-\xi)d\xi \quad (^\forall\phi \in \mathcal{D}_p)$.

We have the inclusion relations :

$$\mathcal{E}'_p(\Omega) \subset \mathcal{D}'_{p,F}(\Omega) \subset \mathcal{D}'_p(\Omega); \mathcal{E}'_p \subset \mathcal{F}_p.$$

If p and $p^* \in \mathcal{M}$ and $p^* \prec p$, then $\mathcal{D}' \subset \mathcal{D}'_{p^*} \subset \mathcal{D}'_p$ algebraically and topologically.

We define $\tau_x u$ of $u \in \mathcal{D}'_p$ by $(\tau_x u)(\phi) = u(\tau_{-x}\phi) \quad (\phi \in \mathcal{D}_p, x \in R^n)$. Then, for each $x \in R^n, \tau_x u \in \mathcal{D}'_p$.

DEFINITION 3.4. A mapping $T : \mathcal{D}'_p \to \mathcal{D}'_p$ is *isometry* if for every $\phi \in \mathcal{D}_p$ there exists $\psi \in \mathcal{D}_p$ such that for every $u, v \in \mathcal{D}'_p$

$$||T(u) - T(v)||_\phi = ||u - v||_\psi.$$

THEOREM 3.5. *Let $p \in \mathcal{M}$ and let $x \in R^n$ be given. Then the mapping $T_x : \mathcal{D}'_p \to \mathcal{D}'_p$ defined by $T_x(u) = \tau_x u$ is an isometry.*

Proof. $||T_x(u)||_\phi = |(\tau_x u)(\phi)| = |u(\tau_{-x}\phi)| = ||u||_{\phi_x}$, where $\phi_x(y) = \phi(x+y)$.

THEOREM 3.6. *Let $p \in \mathcal{M}$ and let $x \in R^n$ be given. Then the mapping $F_x : \mathcal{D}'_p \to \mathcal{D}'_p$ defined by $F_x(u) = e_x u$ is an isometry, where $(e_x u)(\phi) = u(e_x\phi)$ and $(e_x\phi)(y) = e^{ixy}\phi(y)$.*

Proof.

$$||F_x(u)||_\phi = ||e_x u||_\phi = |(e_x u)(\phi)| = ||u||_{e_x\phi}.$$

We define, for $\phi \in \mathcal{D}_p$ and $u \in \mathcal{D}'_p, (\phi * u)(\psi) = u(\check{\phi} * \psi) \quad (^\forall\psi \in \mathcal{D}_p)$, then $\phi * u \in \mathcal{D}'_p$.

THEOREM 3.7. *Let $p \in \mathcal{M}$. If P is a polynomial and $g \in \mathcal{D}_p$, then each of the three mappings*

(1) $F_g : \mathcal{D}_p' \to \mathcal{D}_p'$ by $F_g(u) = gu$
(2) $F_{*g} : \mathcal{D}_p' \to \mathcal{D}_p'$ by $F_{*g}(u) = g * u$
(3) $P(D) : \mathcal{D}_p' \to \mathcal{D}_p'$ by $P(D)(u) = P(D)u$

is continuous and linear.

Proof. (1) $||F_g(u)||_\phi = ||gu||_\phi = |u(gu)| = ||u||_{g\phi}$, where $(gu)(\phi) = u(g\phi) = u(g\phi) \quad (\phi \in \mathcal{D}_p)$. Hence, F_g is an isometry.

(2) $||F_{*g}(u)||_\psi = ||g * u||_\psi = |u(\check{g} * \psi)| = ||u||_{\check{g}*\psi}$. Hence, F_{*g} is an isometry.

(3) Let $P(D) = \sum_{|\alpha| \le m} C_\alpha D_\alpha = \sum_{|\alpha| \le m} C_\alpha (-i)^{-|\alpha|} D^\alpha$.

$$\begin{aligned} ||P(D)u||_\psi = |(P(D)u)(\psi)| &= \Big| \sum_{|\alpha| \le m} C_\alpha u(D^\alpha \phi) \Big| \\ &\le \sum_{|\alpha| \le m} |C_\alpha u(D^\alpha \psi)| \\ &\le \max_{|\alpha| \le m} |C_\alpha| \sum_{|\alpha| \le m} |u(D^\alpha \psi)| \\ &\le C ||u||_{D^\alpha \psi} \end{aligned}$$

for some α with $|\alpha| \le m$ and constant C.

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