Comm. Korean Math. Soc. 10 (1995), No. 1, pp. 115–122

SHARP RESULTS FOR THE MULTIPLICITY OF PERIODIC SOLUTIONS OF A NONLINEAR SUSPENSION BRIDGE EQUATION

Q-HEUNG CHOI AND TACKSUN JUNG

0. Introduction

In this paper we study the multiplicity of periodic solutions of a nonlinear suspension bridge eqaution

$$
\begin{aligned}
&u_{tt} + u_{xxxx} + bu^+ = 1 + \varepsilon h(x,t) \quad \text{in} \quad \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \mathbb{R} \\
&u\left(\pm\frac{\pi}{2}, t\right) = u_{xx}\left(\pm\frac{\pi}{2}, t\right) = 0 \\
&u(x,t) = u(-x,t) = u(x,-t) = u(x, t+\pi).
\end{aligned}
\tag{0.1}
$$

McKenna and Walter [8] has proved that if $3 < b < 15$, (0.1) has at least two solutions. Choi and Jung [3] proved that if $-1 < b < 3$, then (0.1) has a unique solution and that if $3 < b < 15$, then there exists at least three solutions of (0.1) by a variational reduction method, with replacing the condition for $u(x,t)$ in (0.1) by

$$u(x,t) = u(-x,t) = u(x, t+\pi).$$

The purpose of this paper is to show that if $3 < b < 15$, then (0.1) has exactly three solutions.

Received July 27, 1994.

AMS Classification: 34B15, 35Q40.

Key word: Eigenvalue, eigenvector, Leray-Sander degree.

Research supported in part by GARC-KOSEF and Inha University Research Foundation.

1. Preliminaries

We define the differential operator L as follows

$$Lu = u_{tt} + u_{xxxx}.$$

The eigenvalue problem for $u(x,t)$

$$\begin{gathered} Lu = \lambda u \quad \text{in} \quad \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times R, \\ u\left(\pm\frac{\pi}{2}, t\right) = 0, \\ u(x,t) = u(-x,t) = u(x,-t) = u(x,t+\pi), \end{gathered}$$

has infinitely many eigenvalues

$$\lambda_{mn} = (2n+1)^4 - 4m^2 \qquad (m, n = 0, 1, 2, \cdots)$$

and corresponding normalized eigenfunctions ϕ_{mn} $(m, n =\geq 0)$ given by

$$\begin{aligned} \phi_{0n} &= \frac{\sqrt{2}}{\pi} \cos(2n+1)x \quad \text{for} \quad n \geq 0 \\ \phi_{mn} &= \frac{2}{\pi} \cos 2mt \cos(2n+1)x \quad \text{for} \quad m > 0, \ n \geq 0. \end{aligned}$$

We note that all eigenvalues in the interval $(-19, 45)$ are given by

$$\lambda_{20} = -15 < \lambda_{10} = -3 < \lambda_{00} = 1 < \lambda_{41} = 17.$$

Let Q be the square $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and H the Hilbert space defined by

$$H = \{u \in L^2(Q) \mid u \text{ is even in } x \text{ and } t\}.$$

Then kthe set $\{\phi_{mn} \mid m, n = 0, 1, 2, \cdots\}$ is an orthonormal base in H. For simplicity of notation, a weak solution of (0.1) is characterized by

$$Lu + bu^+ = 1 + \varepsilon h(x,t) \quad \text{in} \quad H. \tag{1.1}$$

Now we denote that for given u, $\chi(u)$ is the characteristic function of the positive set of u, i.e.,

$$[\chi(u)](x,t) = \begin{cases} 1, & \text{if } u(x,t) > 0 \\ 0, & \text{if } u(x,t) \leq 0 \end{cases}.$$

Now we consider the operator $(\ )^+$ from $L^p(Q)$ into $L^q(Q)$ which sends u into u^+ for given $p, q > 1$, $q < p$.

We note that if $u \in L^p$ is such that $\mu\{(x,t) \in Q : u(x,t) = 0\} = 0$ for the Lebesque measure μ, then, for given $\varepsilon > 0$, there exists a neighborhood U of u such that if we write

$$v^+ - \chi(u)v = z(u)$$

for $v \in U$, then the function z is a Lipschitz mapping from U into L^q with Lipschitz constant less than or equal to ε (ref. [11]).

We consider the eigenvalue problem

$$-Lu = \nu A u \quad \text{in} \quad H$$

for given $A \in L^{\frac{1}{2}}(Q)$. We note that if $A > 0$ in a set of positive measure, then

$$\text{(1.2)} \qquad \cdots < \nu_{41}(A) < \nu_{00}(A) < 0 < \nu_{10}(A) < \nu_{20}(A) < \cdots .$$

Let us set $A(u) = b\chi(u)$ when

$$\mu\{(x,t) : u(x,t) = 0\} = 0.$$

2. Main result

We state the main result of this paper, which is a sharp result for the multiplicity of solutions of a nonlinear suspension bridge equation.

THEOREM 2.1. *Let $h \in H$ with $\|h\| = 1$ and $3 < b < 15$. Then there exists $\varepsilon_0 > 0$ such that if $|\varepsilon| < \varepsilon_0$, then the equation*

$$\text{(2.1)} \qquad Lu + bu^+ = 1 + \varepsilon h(x,t) \quad \textit{in} \quad H$$

has exactly three solutions.

For the proof of Theorem 2.1 we need some lemmas.

LEMMA 2.1. *For $-1 < b < 15$, the problem*

$$Lu + bu^+ = 0 \quad in \quad H$$

has only the trivial solution $u \equiv 0$ (ref. [8]).

LEMMA 2.2. *Let $h \in H$ with $\|h\| = 1$ and $\alpha > 0$ be given. Then there eixsts $R_0 > 0$ (depending only on h and α) such that for all b with $-1 + \alpha \leq b \leq 15 - \alpha$ and all $\varepsilon \in [-1, 1]$ the solutions of (2.1) satisfy $\|u\| < R_0$* (ref. [8]).

LEMMA 2.3. *Under the assumptions and all the notations of Lemma 2.2*

$$d_{LS}(u - L^{-1}(1 - bu^+ + \varepsilon h), B_R, 0) = 1$$

for all $R \geq R_0$, where d_{LS} denotes the Leray-Schauder degree (ref. [8]).

LEMMA 2.4. *For $b > -1$, the boundary value problem*

$$(2.2) \qquad y^{(4)} + by^+ = 1 \quad in \quad \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \; y\left(\pm\frac{\pi}{2}\right) = y''\left(\pm\frac{\pi}{2}\right) = 0$$

has a unique solution y which is even and positive and satisfies

$$y'\left(-\frac{\pi}{2}\right) > 0 \quad and \quad y'\left(\frac{\pi}{2}\right) < 0$$

(ref. [8]).

LEMMA 2.5. *Let $-1 < b$ with b not an eigenvalue of L. Let $h \in H$ with $\|h\| = 1$ be given. Then there exist $\varepsilon_0 > 0$ (depending on b and h) such that if $|\varepsilon| < \varepsilon_0$, the boundary value problem*

$$Lu + bu^+ = 1 + \varepsilon h(x,t) \quad in \quad H$$

has a positive solution u_0.

Proof. From Lemma 2.4, the boundary value problem

$$y^{(4)} + by^+ = 1 \quad \text{in} \quad \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),$$

$$y\left(\pm\frac{\pi}{2}\right) = y''\left(\pm\frac{\pi}{2}\right) = 0$$

has a unique positive solution y.

We note that if b is not an eigenvalue of L, then the following linear partial defferential equation

$$Lu + bu = \varepsilon h(x,t) \quad \text{in} \quad H$$

has a unique solution u_ε. We can choose sufficiently small $\varepsilon_0 > 0$ (depending on b and h) such that if $|\varepsilon| < \varepsilon_0$ then $u_\varepsilon + y(\text{say } u_0) > 0$ which is a solution of (2.1). So the lemma is proved. □

LEMMA 2.6. *Assume that $3 < b < 15$. Let $h \in H$ with $\|h\| = 1$ be given. Then there exists a small neighborhood U of u_0 and $\varepsilon_2 > 0$ such that if $|\varepsilon| < \varepsilon_2$, then*

$$d_{LS}(u - L^{-1}(1 + bu^+ + \varepsilon h), U(u_0), 0) = -1,$$

where u_0 is a positive solution of (2.1).

Proof. From the Lemma 6 in [8], we have that : If y is the unique positive solution of the boundary value problem in Lemma 2.4, then there eixst $\gamma > 0$, $\varepsilon_1 > 0$ such that

$$d_{LS}(u - L^{-1}(1 + bu^+ + \varepsilon h), B_\gamma(y), 0) = -1$$

for $|\varepsilon| < \varepsilon_1$.

Choose $0 < \varepsilon_2 < \min\{\varepsilon_0, \varepsilon_1\}$ such that if $|\varepsilon| < \varepsilon_2$, then

$$u_\varepsilon + y \in B_\gamma(y)$$

for ε_0 in Lemma 2.5.

Then for $|\varepsilon| < \varepsilon_2$, $B_\gamma(y)$ is the small neighborhood of $u_0 = u_\varepsilon + y$ such that $\partial B_\gamma(y)$ has no solution of the equation

$$u - L^{-1}(1 + bu^+ + \varepsilon h) = 0.$$

Let us take $U(u_0) = B_\gamma(y)$. Then we obtain the desired result. □

Now, we turn attention to the solutions of (2.1) which change sign.

LEMMA 2.7. *Assume that* $3 < b < 15$. *Then if* u *is a solution of* (2.1) *which changes sign, then*

$$\nu_{10}(A(u)) > 1. \tag{2.3}$$

Proof. We know that (2.1) has the positive solution u_0. Writing (2.1) for u and u_0 and subtracting we get

$$-L(u_0 - u) = b(u_0 - u^+). \tag{2.4}$$

If we use the notation $\hat{A} = \dfrac{b(u_0 - u^+)}{u_0 - u}$, then we have

$$0 \le A(u) \le \hat{A} \le b. \tag{2.5}$$

By (2.4), $\nu_{mn}(\hat{A}) = 1$ for some $m, n \ge 0$ and by (2.5), $\nu_{10}(\hat{A}) = 1$.
Since

$$\nu_{10}(A(u)) > \nu_{10}(\hat{A}) = 1,$$

the desired result follows. □

LEMMA 2.8. *Assume that* $3 < b < 15$. *Then if* u_* *is a solution of* (2.1) *which changes sign, then there exists* $\varepsilon_* > 0$ *such that*

$$d_{LS}(u - L^{-1}(1 - bu^+ + \varepsilon h), B_{\varepsilon_*}(u_*), 0) = 1.$$

Proof. Let u_* be a solution of (2.1) which changes sign. Since the solutions of (2.1) are discrete, we can choose small $\varepsilon' > 0$ such that $B_{\varepsilon'}(u_*)$ does not contain the other solutions of (2.1). Let us choose $u \in B_{\varepsilon'}(u_*)$ and set $v = u - u_*$. Then there eixsts $\varepsilon_* < \varepsilon'$ such that the following holds :

$$\begin{aligned} &u - L^{-1}(1 - bu^+ + \varepsilon h) = (u_* + v) - L^{-1}(1 - b(u_* + v)^+ + \varepsilon h) \\ &= v - L^{-1}(bu_*^+ - b(u_* + v)^+) = v - L^{-1}(-b\chi(u_*)v) \\ &= v - L^{-1}(-A(u_*)v). \end{aligned}$$

Thus we have

$$\begin{aligned} &d_{LS}(u - L^{-1}(1 - bu^+ + \varepsilon h), B_{\varepsilon^*}(u_*), 0) \\ &= d_{LS}(v + L^{-1}(A(u_*)v), B_{\varepsilon^*}(0), 0). \end{aligned}$$

The eigenvalues of the operator $v + L^{-1}A(u_*)v$ are connected with the eigenvalues ν of $-L$ by

$$v + L^{-1}A(u_*)v = \rho v \Longleftrightarrow -Lv - A(u_*)v = \rho(-Lv)$$

or $\rho = \dfrac{\nu - 1}{\nu}$. It follows from Lemma 2.7 and (1.2) that there are only positive eigenvalues. Thus the desired degree is +1. So the lemma is proved. □

Proof of Theorem 2.1. The equation (2.1) can be written in the form

$$u - L^{-1}(1 - bu^+ + \varepsilon h) = 0.$$

The degree of $u - L^{-1}(1 - bu^+ + \varepsilon h)$ on a large ball of radius $R > R_0$ is +1 by Lemma 2.3. From Lemma 2.6, the constant sign solution of (2.1) is only the positive solution u_0 and the degree on the small neighborhood $U(u_0)$ is -1. From Lemma 2.8, the degree on the ball $B_{\varepsilon^*}(u_*)$ is +1, if u_* is a solution of (2.1) which changes sign. Choosing $R > R_0$ so that B_R contains all solutions of (2.1), we can conclude that

$$d_{LS}(u - L^{-1}(1 - bu^+ + \varepsilon h), B_R - (U(u_0)), 0) = 2.$$

Since the solutions of (2.1) are discrete and the degree on the ball $B_{\varepsilon^*}(u_*)$ is +1 if $u_*$is a change sign solution of (2.1), there exists exactly two change sign solutions in $B_R \backslash U(u_0)$. Thus there exist exactly three solutions in B_R. □

References

1. A. R. Aftabizadeh, *Existence and uniqueness theorems for fourth order boundary value problems*, J. Math. Anal. Appl. **116** (1986), 415-426.
2. Q. H. Choi and T. Jung, *On periodic solutions of the nonlinear suspension bridge equation*, Diff. Int. Eq. **4** (1991), 383-396.

3. Q. H. Choi, T. Jung, and P. J. McKenna, *The study of a nonlinear suspension bridge equation by a variational reduction method*, Applicable Analysis **50** (1993), 73-92.
4. J. M. Coron, *Periodic solutions of a nonlinear wave equation without assumptions of monotonicity*, Math. Ann. **262** (1983), 273-285.
5. A. C. Lazer and P. J. McKenna, *A symmetry theorem and applications to nonlinear partial differential equations*, J. Diff., Eq. **71** (1988), 95-106.
6. P. J. McKenna, R. Redinger, and W. Walter, *Multiplicity results for asymptotically homogenuous semilinear boundary value problems*, Annali de Matematica pur. appl. (4) **CXL3** (1986), 347-257.
7. P. J. McKenna and W. Walter, *On the multiplicity of the solution set of some nonlinear boundary value problem*, Nonlinear analysis **8** (1984), 893-907.
8. P. J. McKenna and W. Walter, *Nonlinear oscillations in a suspension bridge*, Archive for Rational Mechanics and Analysis **98** (1987), 167-177.
9. L. Nirenberg, *Topics in Nonlinear Functional Analysis*, Courant Inst. Lecture Notes, 1974.
10. J. Schröder, *Operator Inequalities*, Academic Press, 1980.
11. S. Solimini, *Some remarks on the number of solutions of some nonlinear elliptic problems*, Ann. Inst. Henri Poincaré **2** (1985), 143-156.

Department of Mathematics
Inha University
Incheon 402-751, Korea

Department of Mathematics
Kunsan National Univeristy
Kunsan 573-360, Korea