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WEAK COMPATIBLE MAPPINGS OF TYPE (A) AND COMMON FIXED POINTS IN MENGER SPACES

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1. Introduction

The notion of probabilistic metric spaces (or statistical metric spaces) was introduced and studied by Menger [19] which is a generalization of metric space, and the study of these spaces was expanded rapidly with the pioneering works of Schweizer-Sklar [25]-[26]. The theory of probabilistic metric spaces is of fundamental importance in probabilistic function analysis. For the detailed discussions of these spaces and their applications, we refer to [9], [10], [28], [30]-[32], [36] and [39].

Recently, some fixed point theorems in probabilistic metric spaces have been proved by many authors; Bharucha-Reid [1], Bocsan [2], Chang [5], Ćirić [7], Hadžić [11]-[13], Hicks [14], Singh-Pant [33]-[35], Stojaković [37]-[39], Tan [40] and many others ([3], [4], [8], [14], [20] and [42]), and also, some fixed point theorems in Menger spaces have been proved by many authors; Cho-Murthy-Stojaković [6], Dedeic-Sarapa [8], Radu [22]-[24], Stojaković [37]-[39] and others.

Note that every metric space is a probabilistic metric space and hence we can use many results in probabilistic metric spaces to prove some fixed point theorems in metric spaces and Banach spaces.

Recently, Jungck [15] generalized the Banach contraction principle by using the concept of compatible mappings on metric spaces. Of course, any weakly commuting mappings are compatible mappings but the converses are not true.

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The existence of fixed points for compatible mappings on metric spaces and probabilistic metric spaces is shown by Jungck [15]-[17], Mishra [20] and Sessa-Rhoades-Khan [29].

Quite recently, Jungck-Murthy-Cho [18] and Cho-Murthy-Stojaković [6] introduced the concept of compatible mappings of type (A) on metric spaces and Menger spaces respectively, and proved the existence of fixed points of these mappings in metric and Menger spaces, respectively.

In this paper, we introduce the concept of weak compatible mappings of type (A) on Menger spaces, which is equivalent to the concepts of compatible mappings and compatible mappings of type (A) under some conditions, and prove some common fixed point theorems for weak compatible mappings of type (A) on Menger spaces. Our results generalize and improve results of Cho-Murthy-Stojaković [6].

2. Preliminaries

Let R denote the set of reals and R^+ the nonnegative reals. A mapping $\mathcal{F} : R \to R^+$ is called a *distribution function* if it is nondecreasing left continuous with $\inf \mathcal{F} = 0$ and $\sup \mathcal{F} = 1$. We will denote $\mathcal{L}$ by the set of all distribution functions.

A *probabilistic metric space* (briefly, a PM-space) is a pair $(X, \mathcal{F})$, where X is a nonempty set and $\mathcal{F}$ is a mapping from $X \times X$ to $\mathcal{L}$. For $(u, v) \in X \times X$, the distribution function $\mathcal{F}(u, v)$ is denoted by $F_{u,v}$. The functions $F_{u,v}$ are assumed to satisfy the following conditions:

(P1) $F_{u,v}(x) = 1$ for every $x > 0$ if and only if $u = v$,

(P2) $F_{u,v}(0) = 0$ for every $u, v \in X$,

(P3) $F_{u,v}(x) = F_{v,u}(x)$ for every $u, v \in X$,

(P4) If $F_{u,v}(x) = 1$ and $F_{v,w}(y) = 1$, then $F_{u,w}(x + y) = 1$ for every $u, v, w \in X$.

In a metric space (X, d), the metric d induces a mapping $\mathcal{F} : X \times X \to \mathcal{L}$ such that

$$\mathcal{F}(u, v)(x) = F_{u,v}(x) = H(x - d(u, v))$$

for every $u, v \in X$ and $x \in R$, where H is a distribution function defined by

$$H(x) = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0. \end{cases}$$

A function $t : [0,1] \times [0,1] \to [0,1]$ is called a *T-norm* if it satisfies the following conditions :

(t1) $t(a,1) = a$ for every $a \in [0,1]$ and $t(0,0) = 0$,
(t2) $t(a,b) = t(b,a)$ for every $a,b \in [0,1]$,
(t3) If $c \geq a$ and $d \geq b$, then $t(c,d) \geq t(a,b)$,
(t4) $t(t(a,b),c) = t(a,t(b,c))$ for every $a,b,c \in [0,1]$.

A *Menger space* is a triple $(X, \mathcal{F}, t)$, where $(X, \mathcal{F})$ is a PM-space and t is a T-norm with the following condition:

(P5) $F_{u,w}(x+y) \geq t(F_{u,v}(x), F_{v,w}(y))$ for every $u,v,w \in X$ and $x,y \in R^+$.

The concept of neighbourhoods in PM-spaces was introduced by Schweizer-Sklar [25]. If $u \in X$, $\epsilon > 0$ and $\lambda \in (0,1)$, then an (ϵ,λ)-neighbourhood of u, denoted by $U_u(\epsilon,\lambda)$, is defined by

$$U_u(\epsilon,\lambda) = \{v \in X : F_{u,v}(\epsilon) > 1 - \lambda\}.$$

If $(X, \mathcal{F}, t)$ is a Menger space with the continuous T-norm t, then the family

$$\{U_u(\epsilon,\lambda) : u \in X, \epsilon > 0, \lambda \in (0,1)\}$$

of neighbourhoods induces a Hausdorff topology on X and if $\sup_{a<1} t(a,a) = 1$, it is metrizable.

An important T-norm is the T-norm $t(a,b) = \min\{a,b\}$ for all $a,b \in [0,1]$ and this is the unique T-norm such that $t(a,a) \geq a$ for every $a \in [0,1]$. Indeed, if it satisfies this condition, we have

$$\begin{aligned}\min\{a,b\} &\leq t(\min\{a,b\},\min\{a,b\}) \leq t(a,b) \\ &\leq t(\min\{a,b\},1) = \min\{a,b\}.\end{aligned}$$

Therefore, $t = \min$.

In the sequel, we need the following definitions and theorems are well-known ([23]):

DEFINITION 2.1. Let $(X, \mathcal{F}, t)$ be a Menger space. A mapping S from X into itself is said to be *continuous* at a point $p \in X$ if for every $\epsilon > 0$ and $\lambda > 0$, there exist $\epsilon_1 > 0$ and $\lambda_1 > 0$ such that if $q \in U_p(\epsilon_1, \lambda_1)$, then $Sq \in U_{Sp}(\epsilon,\lambda)$, that is, if $F_{p,q}(\epsilon_1) > 1 - \lambda_1$, then $F_{Sp,Sq}(\epsilon) > 1 - \lambda$.

DEFINITION 2.2. Let $(X, \mathcal{F}, t)$ be a Menger space with the continuous T-norm t. A sequence $\{p_n\}$ in X is said to be *convergent* to a point $p \in X$ if for every $\epsilon > 0$ and $\lambda > 0$, there exists an integer $N = N(\epsilon, \lambda)$ such that $p_n \in U_p(\epsilon, \lambda)$ for all $n \geq N$, or equivalently, $F_{p,p_n}(\epsilon) > 1 - \lambda$ for all $n \geq N$. We write $p_n \to p$ as $n \to \infty$ or $\lim_{n\to\infty} p_n = p$.

Since the (ϵ, λ)-topology on Menger space $(X, \mathcal{F}, t)$ satisfies the first axiom of the countability, we have the following:

THEOREM 2.1. *Let $(X, \mathcal{F}, t)$ be a Menger space with the continuous T-norm t and S be a mapping from X into itself. Then S is continuous at a point p if and only if for every sequence $\{p_n\}$ converging to p, the sequence $\{Sp_n\}$ converges to the point Sp.*

THEOREM 2.2. *Let $(X, \mathcal{F}, t)$ be a Menger space with the continuous T-norm t. Then $\mathcal{F}$ is a lower semi-continuous function of points in X, that is, for any fixed $x \in R^+$, if $q_n \to q$ and $p_n \to p$ as $n \to \infty$, then*

$$\lim_{n\to\infty} \inf F_{p_n,q_n}(x) = F_{p,q}(x).$$

DEFINITION 2.3. Let $(X, \mathcal{F}, t)$ be a Menger space with the continuous T-norm t. A sequence $\{p_n\}$ of points in X is said to be a *Cauchy sequence* if for every $\epsilon > 0$ and $\lambda > 0$, there exists an integer $N = N(\epsilon, \lambda) > 0$ such that $F_{p_n,p_m}(\epsilon) > 1 - \lambda$ for all $m, n \geq N$.

DEFINITION 2.4. A Menger space $(X, \mathcal{F}, t)$ with the continuous T-norm t is said to be *complete* if every Cauchy sequence in X converges to a point in X.

The following theorems establish the relations between metric spaces and Menger spaces. It is well known that every metric space (X, d) is a Menger space $(X, \mathcal{F}, \min)$, where the mapping $F_{x,y}$ is defined by $F_{x,y}(\epsilon) = H(\epsilon - d(x, y))$. The space $(X, \mathcal{F}, \min)$ is called the induced Menger space.

THEOREM 2.3. *Let t be a T-norm defined by $t(a, b) = \min\{a, b\}$. Then the induced Menger space $(X, \mathcal{F}, t)$ is complete if a metric space (X, d) is complete.*

THEOREM 2.4. *Let $(X, \mathcal{F}, t)$ be an induced Menger space by the metric d. Let $\{p_n\}$ be a sequence in X and S be a mapping from X into itself. Then for every $\epsilon > 0$ and $\lambda > 0$, $F_{p_n,p}(\epsilon) > 1 - \lambda$ if and only if there exists an integer N such that $d(p_n, p) < \epsilon$ for all $n \geq N$, and S is continuous at p in the sense of the Menger space if and only if S is continuous at p in the sense of the metric space.*

3. Compatible mappings of type (A)

In this section, motivated by the concept of compatible mappings and compatible mappings of type (A) on metric spaces and PM-spaces ([15], [6] and [20]), we introduce the concept of weak compatible mappings of type (A) on Menger spaces. In metric spaces and Menger spaces, the concepts of compatible mappings and compatible mappings of type (A) are equivalent under some conditions ([18] and [6]).

DEFINITION 3.1. Let $(X, \mathcal{F}, t)$ be a Menger space such that the T-norm t is continuous and S, T be mappings from X into itself. S and T are said to be *compatible* if

$$\lim_{n\to\infty} F_{STx_n,TSx_n}(x) = 1$$

for all $x > 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$.

DEFINITION 3.2. Let $(X, \mathcal{F}, t)$ be a Menger space such that the T-norm t is continuous and S, T be mappings from X into itself. S and T are said to be *compatible of type* (A) if

$$\lim_{n\to\infty} F_{TSx_n,SSx_n}(x) = 1 \quad \text{and} \quad \lim_{n\to\infty} F_{STx_n,TTx_n}(x) = 1$$

for all $x > 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$.

DEFINITION 3.3. Let $(X, \mathcal{F}, t)$ be a Menger space such that the T-norm t is continuous and S, T be mappings from X into itself. S and T are said to be *weak compatible of type* (A) if

$$\lim_{n\to\infty} F_{STx_n,TTx_n}(x) \geq \lim_{n\to\infty} F_{TSx_n,TTx_n}(x)$$

and

$$\lim_{n\to\infty} F_{TSx_n,SSx_n}(x) \geq \lim_{n\to\infty} F_{STx_n,SSx_n}(x)$$

for all $x > 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$.

First, the following Propositions 3.1 and 3.3 show that Definitions 3.1 and 3.2 are equivalent under some conditions ([6]):

PROPOSITION 3.1. *Let $(X, \mathcal{F}, t)$ be a Menger space such that the T-norm t is continuous and $t(x,x) \geq x$ for all $x \in [0,1]$, and $S, T : X \to X$ be continuous mappings. If S and T are compatible, then they are compatible of type (A).*

PROPOSITION 3.2. *Let $(X, \mathcal{F}, t)$ be a Menger space such that the T-norm t is continuous and $t(x,x) \geq x$ for all $x \in [0,1]$, and let $S, T : X \to X$ be compatible mappings of type (A). If one of S and T is continuous, then S and T are compatible.*

From Propositions 3.1 and 3.2, we have:

PROPOSITION 3.3. *Let $(X, \mathcal{F}, t)$ be a Menger space such that the T-norm t is continuous and $t(x,x) \geq x$ for all $x \in [0,1]$, and $S, T : X \to X$ be continuous mappings. Then S and T are compatible if and only if they are compatible of type (A).*

REMARK 1. In [18], one may find examples which says that Proposition 3.3 is not true if S and T are not continuous on X.

The following propositions show that Definitions 3.1~3.3 are equivalent under some conditions, but first we have:

PROPOSITION 3.4. *Let $(X, \mathcal{F}, t)$ be a Menger space such that the T-norm t is continuous and $t(x,x) \geq x$ for all $x \in [0,1]$, and $S, T : X \to X$ be continuous mappings. Then S and T are weak compatible of type (A) if they are compatible of type (A).*

Proof. Suppose that S and T are compatible mappings of type (A), then we have

$$1 = \lim_{n\to\infty} F_{STx_n,TTx_n}(x) \geq \lim_{n\to\infty} F_{TSx_n,TTx_n}(x)$$

and

$$1 = \lim_{n\to\infty} F_{TSx_n,SSx_n}(x) \geq \lim_{n\to\infty} F_{STx_n,SSx_n}(x).$$

This completes the proof.

PROPOSITION 3.5. *Let $(X, \mathcal{F}, t)$ be a Menger space such that the T-norm t is continuous and $t(x,x) \geq x$ for all $x \in [0,1]$, and $S, T : X \to X$ be continuous mappings. If S and T are weak compatible of type (A), then they are compatible of type (A).*

Proof. Let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$. Since S and T are continuous, we have

$$\lim_{n\to\infty} F_{STx_n,TTx_n}(\epsilon) \geq \lim_{n\to\infty} F_{TSx_n,TTx_n}(\epsilon) = \lim_{n\to\infty} F_{Tz,Tz}(\epsilon) = 1$$

and

$$\lim_{n\to\infty} F_{TSx_n,SSx_n}(\epsilon) \geq \lim_{n\to\infty} F_{STx_n,SSx_n}(\epsilon) = \lim F_{Sz,Sz} = 1$$

for all $\epsilon > 0$. Therefore, S and T are compatible mappings of type (A). This completes the proof.

PROPOSITION 3.6. *Let $(X, \mathcal{F}, t)$ be a Menger space such that the T-norm t is continuous and $t(x,x) \geq x$ for all $x \in [0,1]$, and $S, T : X \to X$ be weak compatible of type (A). If one of S and T is continuous, then S and T are compatible.*

Proof. Let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$.

Suppose that S and T are weak compatible of type (A). Assume, without loss of generality, that S is continuous. Then $\lim_{n\to\infty} STx_n = Sz = \lim_{n\to\infty} SSx_n$ and so, for positive reals ϵ and λ, there exists an integer $M(\epsilon, \lambda)$ such that

$$F_{STx_n,Sz}(\epsilon/2) > 1 - \lambda \quad \text{and} \quad F_{SSx_n,Sz}(\epsilon/2) > 1 - \lambda$$

for all $n \geq M(\epsilon, \lambda)$. Further, since S and T are weak compatible of type (A), we have

$$\lim_{n\to\infty} F_{TSx_n,SSx_n}(\epsilon/2) \geq \lim_{n\to\infty} F_{STx_n,SSx_n}(\epsilon/2) = 1.$$

By (P5) and (P3), we have

$$F_{STx_n,TSx_n}(\epsilon/2) \geq t\big(F_{STx_n,SSx_n}(\epsilon/2), F_{SSx_n,TSx_n}(\epsilon/2)\big),$$

it follows that $\lim_{n\to\infty} F_{STx_n,TSx_n}(\epsilon) = 1$. This completes the proof.

As a direct consequence of Propositions 3.1, 3.4 and 3.6, we have the following:

PROPOSITION 3.7. *Let $(X, \mathcal{F}, t)$ be a Menger space such that the T-norm t is continuous and $t(x,x) \geq x$ for all $x \in [0,1]$, and $S, T : X \to X$ be continuous mappings. Then S and T are compatible if and only if they are weak compatible of type (A).*

By unifying Propositions 3.4, 3.5 and 3.7, we have the following:

PROPOSITION 3.8. *Let $(X, \mathcal{F}, t)$ be a Menger space such that the T-norm t is continuous and $t(x,x) \geq x$ for all $x \in [0,1]$, and $S, T : X \to X$ be continuous mappings. Then*

(1) *S and T are compatible type (A) if and only if they are weak compatible of type (A).*
(2) *S and T are compatible if and only if they are weak compatible of type (A).*

REMARK 2. In [21], one can find examples which says that Propositions 3.5 and 3.8 (2) is not true if S and T are not continuous on X.

Next, we give several properties of weak compatible mappings of type (A) on a Menger spaces for our main theorems:

PROPOSITION 3.9. *Let $(X, \mathcal{F}, t)$ be a Menger space such that the T-norm t is continuous and $t(x,x) \geq t$ for all $x \in [0,1]$, and $S, T : X \to X$ be mappings. If S and T are weak compatible of type (A) and $Sz = Tz$ for some $z \in X$, then $SSz = STz = TSz = TTz$.*

Proof. Suppose that $\{x_n\}$ is a sequence in X defined by $x_n = z$, $n = 1, 2, \cdots$, for some $z \in X$ and $Sz = Tz$. Then we have Sx_n, $Tx_n \to Sz$ as $n \to \infty$. Since S and T are weak compatible of type (A), for every $\epsilon > 0$,

$$\begin{aligned} F_{STz,TTz}(\epsilon) &= \lim_{n\to\infty} F_{STx_n,TTx_n}(\epsilon) \\ &\geq \lim_{n\to\infty} F_{TSx_n,TTx_n}(\epsilon) = F_{TSz,TTz}(\epsilon) = 1. \end{aligned}$$

Hence, we have $STz = TTz$. Therefore, we have $STz = SSz = TTz = TSz$ since $Tz = Sz$. This completes the proof.

PROPOSITION 3.10. *Let $(X, \mathcal{F}, t)$ be a Menger space such that the T-norm t is continuous and $t(x,x) \geq x$ for all $x \in [0,1]$, and $S, T : X \to X$ be mappings. Let S and T be weak compatible mappings of type (A) and $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$. Then we have*

(1) *$\lim_{n\to\infty} TSx_n = Sz$ if S is continuous.*
(2) *$\lim_{n\to\infty} STx_n = Tz$ if T is continuous.*
(3) *$STz = TSz$ and $Sz = Tz$ if S and T are continuous.*

Proof. (1) Suppose that S is continuous at z. Since $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$, we have $SSx_n \to Sz$ as $n \to \infty$, or equivalently, for any positive reals ϵ and λ, there exists an integer $M(\epsilon, \lambda)$ such that $F_{SSx_n,Sz}(\epsilon/2) > 1 - \lambda$ for all $n \geq M(\epsilon, \lambda)$. Since S and T are weak compatible of type (A), for every $\epsilon > 0$,

$$\lim_{n\to\infty} F_{TSx_n,SSx_n}(\epsilon/2) \geq \lim_{n\to\infty} F_{STx_n,SSx_n}(\epsilon/2)$$

and so we have

$$F_{TSx_n,Sz}(\epsilon) \geq t\big(F_{TSx_n,SSx_n}(\epsilon/2), F_{SSx_n,Sz}(\epsilon/2)\big) > 1 - \lambda$$

for all $n \geq M(\epsilon, \lambda)$. Now, we have

$$\begin{aligned}\lim_{n\to\infty} F_{TSx_n,Sz}(\epsilon) &\geq t\big(\lim_{n\to\infty} F_{TSx_n,SSx_n}(\epsilon/2), \lim_{n\to\infty} F_{SSx_n,Sz}(\epsilon/2)\big)\\ &\geq t\big(\lim_{n\to\infty} F_{STx_n,SSx_n}(\epsilon/2), \lim_{n\to\infty} F_{SSx_n,Sz}(\epsilon/2)\big)\\ &= t\big(F_{Sz,Sz}(\epsilon/2), F_{Sz,Sz}(\epsilon/2)\big)\\ &= t(1,1) = 1.\end{aligned}$$

Therefore, $\lim_{n\to\infty} TSx_n = Sz$ This completes the proof.

(2) The proof of $\lim_{n\to\infty} STx_n = T_z$ follows on the similar lines as argued in (1).

(3) Suppose that $S, T : X \to X$ are continuous at z. Since $Tx_n \to z$ as $n \to \infty$ and S is continuous at z, by (1), $TSx_n \to Sz$ as $n \to \infty$. On the other hand, since $Sx_n \to z$ as $n \to \infty$ and T is also continuous at z, $TSx_n \to Tz$. Thus, we have $Sz = Tz$ by the uniqueness of the limit and so, by Proposition 3.9, $TSz = STz$. This completes the proof.

4. Common fixed point theorems

Before proving our main theorems, we need the following lemma ([26] and [34]):

LEMMA 4.1. *Let $\{x_n\}$ be a sequence in a Menger space $(X, \mathcal{F}, t)$, where t is a continuous T-norm and $t(x,x) \geq x$ for all $x \in [0,1]$. If there exists a constant $k \in (0,1)$ such that*

$$F_{x_n, x_{n+1}}(kx) \geq F_{x_{n-1}, x_n}(x)$$

for all $x > 0$ and $n \in N$, then $\{x_n\}$ is a Cauchy sequence in X.

REMARK 3. In Propositions 3.1, 3.5 and Lemma 4.1, the conditions "the T-norm t is continuous and $t(x,x) \geq x$ for all $x \in [0,1]$" can be replaced by "$t(x,y) = \min\{x,y\}$ for all $x, y \in [0,1]$". In fact, since $t(a,1) = a$ and $t(1,b) = b$ for all $a, b \in [0,1]$, we have

$$t(a,b) \leq \min\{t(a,1), t(1,b)\} = \min\{a,b\}$$

for all $a, b \in [0,1]$. On the other hand, we have

$$t(a,b) \geq t(\min\{a,b\}, \min\{a,b\}) \geq \min\{a,b\}$$

for all $a, b \in [0,1]$, which implies $t(a,b) = \min\{a,b\}$.

Now, we are ready to prove our main theorems:

THEOREM 4.2. *Let $(X, \mathcal{F}, t)$ be a complete Menger space with $t(x,y) = \min\{x,y\}$ for all $x, y \in [0,1]$ and A, B, S, T be mappings from X into itself such that*

(4.1) *$A(X) \subset T(X)$ and $B(X) \subset S(X)$,*

(4.2) *the pairs A, S and B, T are weak compatible of type (A),*

(4.3) *one of A, B, S and T is continuous,*

$$\begin{aligned}\left[F_{Au,Bv}(kx)\right]^2 \geq \min\{ &\left[F_{Su,Tv}(x)\right]^2, F_{Su,Au}(x) \cdot F_{Tv,Bv}(x), \\ &F_{Su,Tv}(x) \cdot F_{Su,Au}(x), F_{Su,Tv}(x) \cdot F_{Tv,Bv}(x), \\ &F_{Su,Tv}(2x) \cdot F_{Su,Bv}(x), F_{Su,Tv}(x) \cdot F_{Tv,Au}(x), \\ &F_{Su,Bv}(2x) \cdot F_{Tv,Au}(x), F_{Sx,Ax}(x) \cdot F_{Tv,Au}(x), \\ &F_{Su,Bv}(2x) \cdot F_{Tv,Bv}(x)\},\end{aligned} \tag{4.4}$$

for all $u, v \in X$ and $x \geq 0$, where $k \in (0,1)$.
Then A, B, S and T have a unique common fixed point in X.

Proof. By (4.1), since $A(X) \subset T(X)$, for any arbitrary $x_0 \in X$, there exists a point $x_1 \in X$ such that $Ax_0 = Tx_1$. Since $B(X) \subset S(X)$, for this point x_1, we can choose a point $x_2 \in X$ such that $Bx_1 = Sx_2$ and so on. Inductively, we can define a sequence $\{y_n\}$ in X such that

$$(4.5) \qquad y_{2n} = Tx_{2n+1} = Ax_{2n} \quad \text{and} \quad y_{2n+1} = Sx_{2n+2} = Bx_{2n+1}$$

for $n = 0, 1, 2, \cdots$.

Now, we shall prove $F_{y_{2n}, y_{2n+1}}(kx) \geq F_{y_{2n-1}, y_{2n}}(x)$ for all $x > 0$, where $k \in (0,1)$. Suppose that $F_{y_{2n}, y_{2n+1}}(kx) < F_{y_{2n-1}, y_{2n}}(x)$. Then by using (4.4) and $F_{y_{2n}, y_{2n+1}}(kx) \leq F_{y_{2n}, y_{2n+1}}(x)$, we have

$$\begin{aligned}
&\left[F_{y_{2n}, y_{2n+1}}(kx)\right]^2 \\
&= \left[F_{Ax_{2n}, Bx_{2n+1}}(kx)\right]^2 \\
&\geq \min\{\left[F_{y_{2n-1}, y_{2n}}(x)\right]^2, F_{y_{2n-1}, y_{2n}}(x) \cdot F_{y_{2n}, y_{2n+1}}(x), \\
&\quad F_{y_{2n-1}, y_{2n}}(x) \cdot F_{y_{2n-1}, y_{2n}}(x), F_{y_{2n-1}, y_{2n}}(x) \cdot F_{y_{2n}, y_{2n+1}}(x), \\
&\quad F_{y_{2n-1}, y_{2n}}(x) \cdot F_{y_{2n-1}, y_{2n+1}}(2x), F_{y_{2n-1}, y_{2n}}(x) \cdot F_{y_{2n}, y_{2n}}(x), \\
&\quad F_{y_{2n-1}, y_{2n+1}}(2x) \cdot F_{y_{2n}, y_{2n}}(x), F_{y_{2n-1}, y_{2n}}(x) \cdot F_{y_{2n}, y_{2n}}(x), \\
&\quad F_{y_{2n-1}, y_{2n+1}}(2x) \cdot F_{y_{2n}, y_{2n+1}}(x)\} \\
&\geq \min\{\left[F_{y_{2n-1}, y_{2n}}(x)\right]^2, F_{y_{2n-1}, y_{2n}}(x) \cdot F_{y_{2n}, y_{2n+1}}(x), \\
&\quad \left[F_{y_{2n-1}, y_{2n}}(x)\right]^2, F_{y_{2n-1}, y_{2n}}(x) \cdot F_{y_{2n}, y_{2n+1}}(x), \\
&\quad F_{y_{2n-1}, y_{2n}}(x) \cdot t\big(F_{y_{2n-1}, y_{2n}}(x), F_{y_{2n}, y_{2n+1}}(x)\big), F_{y_{2n-1}, y_{2n}}(x), \\
&\quad t\big(F_{y_{2n-1}, y_{2n}}(x), F_{y_{2n}, y_{2n+1}}(x)\big), F_{y_{2n-1}, y_{2n}}(x), \\
&\quad t\big(F_{y_{2n-1}, y_{2n}}(x), F_{y_{2n}, y_{2n+1}}(x)\big) \cdot F_{y_{2n}, y_{2n+1}}(x)\} \\
&> \min\{\left[F_{y_{2n}, y_{2n+1}}(kx)\right]^2, \left[F_{y_{2n}, y_{2n+1}}(kx)\right]^2, \left[F_{y_{2n}, y_{2n+1}}(kx)\right]^2, \\
&\quad \left[F_{y_{2n}, y_{2n+1}}(kx)\right]^2, \left[F_{y_{2n}, y_{2n+1}}(kx)\right]^2, F_{y_{2n}, y_{2n+1}}(kx), \\
&\quad F_{y_{2n}, y_{2n+1}}(kx), F_{y_{2n}, y_{2n+1}}(kx), \left[F_{y_{2n}, y_{2n+1}}(kx)\right]^2 \\
&= \left[F_{y_{2n}, y_{2n+1}}(kx)\right]^2,
\end{aligned}$$

which is a contradiction. Thus, we have $F_{y_{2n},y_{2n+1}}(kx) \geq F_{y_{2n-1},y_{2n}}(x)$. Similarly, we have also $F_{y_{2n+1},y_{2n+2}}(kx) \geq F_{y_{2n},y_{2n+1}}(x)$. Therefore, for every $n \in N$, $F_{y_n,y_{n+1}}(kx) \geq F_{y_{n-1},y_n}(x)$. Therefore, by Lemma 4.1, $\{y_n\}$ is a Cauchy sequence in X. Since the Menger space $(X, \mathcal{F}, t)$ is complete, $\{y_n\}$ converges to a point z in X, and the subsequences $\{Ax_{2n}\}$, $\{Bx_{2n+1}\}$, $\{Sx_{2n}\}$, $\{Tx_{2n+1}\}$ of $\{y_n\}$ also converge to z.

Now, suppose that A is continuous. Since A and S are weak compatible of type (A), it follows from Proposition 3.10 that

$$AAx_{2n} \text{ and } SAx_{2n} \to Az \quad \text{as } n \to \infty.$$

By (4.4), we have

$$\begin{aligned}
&\big[F_{AAx_{2n},Bx_{2n+1}}(x)\big]^2 \\
&\geq \min\big\{\big[F_{SAx_{2n},Tx_{2n+1}}(x)\big]^2, F_{SAx_{2n},AAx_{2n}}(x)\cdot F_{Tx_{2n+1},Bx_{2n+1}}(x), \\
&\quad F_{SAx_{2n}(x),Tx_{2n+1}}(x)\cdot F_{SAx_{2n},AAx_{2n}}(x), \\
&\quad F_{SAx_{2n},Tx_{2n+1}}(x)\cdot F_{Tx_{2n+1},Bx_{2n+1}}(x), \\
&\quad F_{SAx_{2n},Tx_{2n+1}}(x)\cdot F_{SAx_{2n},Bx_{2n+1}}(2x), \\
&\quad F_{SAx_{2n},Tx_{2n+1}}(x)\cdot F_{Tx_{2n+1},AAx_{2n}}(x), \\
&\quad F_{SAx_{2n},Bx_{2n+1}}(2x)\cdot F_{Tx_{2n+1},AAx_{2n}}(x), \\
&\quad F_{SAx_{2n},AAx_{2n}}(x)\cdot F_{Tx_{2n+1},AAx_{2n}}(x), \\
&\quad F_{SAx_{2n},Bx_{2n+1}}(2x)\cdot F_{Tx_{2n+1},Bx_{2n+1}}(x))\big\}.
\end{aligned}$$

Taking $n \to \infty$, we have

$$\begin{aligned}
&\big[F_{Az,z}(kx)\big]^2 \\
&\geq \min\big\{\big[F_{Az,z}(x)\big]^2, F_{Az,Az}(x)\cdot F_{z,z}(x), F_{Az,z}(x)\cdot F_{Az,Az}(x), \\
&\quad F_{Az,z}(x)\cdot F_{z,z}(x), F_{Az,z}(x)\cdot F_{Az,z}(2x), \\
&\quad F_{Az,z}(x)\cdot F_{z,Az}(x), F_{Az,z}(2x)\cdot F_{z,Az}(x), F_{Az,Az}(x)\cdot F_{z,Az}(x), \\
&\quad F_{Az,z}(2x)\cdot F_{z,z}(x)\big\} \\
&= \big[F_{Az,z}(x)\big]^2,
\end{aligned}$$

which is a contradiction. Thus we have $Az = z$. Since $A(X) \subset T(X)$, there exists a point $u \in X$ such that $z = Az = Tp$. Again by (3.2), we have

$$\begin{aligned}
&\big[F_{AAx_{2n},Bp}(kx)\big]^2 \\
&\geq \min\big\{\big[F_{SAx_{2n},Tp}(x)\big]^2, F_{SAx_{2n},AAx_{2n}}(x)\cdot F_{Tp,Bp}(x), \\
&\quad F_{SAx_{2n},Tp}(x)\cdot F_{SAx_{2n},AAx_{2n}}(x), F_{SAx_{2n},Tp}(x)\cdot F_{Tp,Bp}(x), \\
&\quad F_{SAx_{2n},Tp}(x)\cdot F_{SAx_{2n},Bp}(2x), F_{SAx_{2n},Tp}(x)\cdot F_{Tp,AAx_{2n}}(x), \\
&\quad F_{SAx_{2n},Bp}(2x)\cdot F_{Tp,AAx_{2n}}(x), F_{SAx_{2n},AAx_{2n}}(x)\cdot F_{Tp,AAx_{2n}}(x), \\
&\quad F_{SAx_{2n},Bp}(2x)\cdot F_{Tp,Bp}(x)\big\}.
\end{aligned}$$

By letting $n \to \infty$, we obtain

$$\begin{aligned}
\big[F_{z,Bp}(kx)\big]^2 &\geq \min\big\{\big[F_{Az,Tp}(x)\big]^2, F_{Az,Az}(x)\cdot F_{Tp,Bp}(x), \\
&\qquad F_{Az,Tp}(x)\cdot F_{Az,Az}(x), F_{Az,Tp}(x)\cdot F_{Tp,Bp}(x), \\
&\qquad F_{Az,Tp}(x)\cdot F_{Az,Bp}(2x), F_{Az,Tp}(x)\cdot F_{Tp,Az}(x), \\
&\qquad F_{Az,Bp}(2x)\cdot F_{Tp,Az}(x), F_{Az,Az}(x)\cdot F_{Tp,Az}(x), \\
&\qquad F_{Az,Bp}(2x)\cdot F_{Tp,Bp}(x)\big\} \\
&\geq \big[F_{z,Bp}(x)\big]^2,
\end{aligned}$$

which implies that $z = Bp$. Since B and T are weak compatible of type (A) and $Tp = Bp = z$, by Proposition 3.9, $TBp = BTp$ and hence $Tz = TBp = BTp = Bz$. Moreover, by (4.4), we have

$$\begin{aligned}
&\big[F_{Ax_{2n},Bz}(kx)\big]^2 \\
&\leq \min\big\{\big[F_{Sx_{2n},Tz}(x)\big]^2, F_{Sx_{2n},Ax_{2n}}(x)\cdot F_{Tz,Bz}(x), \\
&\quad F_{Sx_{2n},Tz}(x)\cdot F_{Sx_{2n},Ax_{2n}}(x), F_{Sx_{2n},Tz}(x)\cdot F_{Tz,Bz}(x), \\
&\quad F_{Sx_{2n},Tz}(x)\cdot F_{Sx_{2n},Bz}(2x), F_{Sx_{2n},Tz}(x)\cdot F_{Tz,Ax_{2n}}(x), \\
&\quad F_{Sx_{2n},Bz}(2x)\cdot F_{Tz,Ax_{2n}}(x), F_{Sx_{2n},Ax_{2n}}(x)\cdot F_{Tz,Ax_{2n}}(x), \\
&\quad F_{Sx_{2n},Bz}(2x)\cdot F_{Tz,Bz}(x)\big\}.
\end{aligned}$$

By letting $n \to \infty$, we obtain

$$\begin{aligned}\left[F_{z,Bz}(kx)\right]^2 \geq \min\{&\left[F_{z,Tz}(x)\right]^2, F_{z,z}(x)\cdot F_{Tz,Bz}(x),\\ &F_{z,Tz}(x)\cdot F_{z,z}(x), F_{z,Tz}(x)\cdot F_{Tz,Bz}(x),\\ &F_{z,Tz}(x)\cdot F_{z,Bz}(2x), F_{z,Tz}(x)\cdot F_{Tz,z}(x),\\ &F_{z,Bz}(2x)\cdot F_{Tz,z}(x), F_{z,z}(x)\cdot F_{Tz,z}(x),\\ &F_{z,Bz}(2x)\cdot F_{Tz,Bz}(x)\}\\ = &\left[F_{z,Bz}(x)\right]^2,\end{aligned}$$

which means that $z = Bz$. Since $B(X) \subset S(X)$, there exists a point $q \in X$ such that $z = Bz = Sq$. By using (4.4), we have

$$\begin{aligned}\left[F_{Aq,z}(kx)\right]^2 &= \left[F_{Aq,Bz}(kx)\right]^2\\ &\geq \min\{\left[F_{Sq,Tz}(x)\right]^2, F_{Sq,Aq}(x)\cdot F_{Tz,Bz}(x),\\ &\quad F_{Sq,Tz}(x)\cdot F_{Sq,Aq}(x), F_{Sq,Tz}(x)\cdot F_{Tz,Bz}(x),\\ &\quad F_{Sq,Tz}(x)\cdot F_{Sq,Bz}(2x), F_{Sq,Tz}(x)\cdot F_{Tz,Aq}(x),\\ &\quad F_{Sq,Bz}(2x)\cdot F_{Tz,Aq}(x), F_{Sq,Aq}(x)\cdot F_{Tz,Aq}(x),\\ &\quad F_{Sq,Bz}(2x)\cdot F_{Tz,Bz}(x)\}\\ &= \min\{\left[F_{z,z}(x)\right]^2, F_{z,Aq}(x)\cdot F_{z,z}(x), F_{z,z}(x)\cdot F_{z,Aq}(x),\\ &\quad F_{z,z}(x)\cdot F_{z,z}(x), F_{z,z}(x)\cdot F_{z,z}(2x), F_{z,z}(x)\cdot F_{z,Aq}(x),\\ &\quad F_{z,z}(2x)\cdot F_{z,Aq}(x), F_{z,Aq}(x)\cdot F_{z,Aq}(x),\\ &\quad F_{z,z}(2x)\cdot F_{z,z}(x)\}\\ &= \left[F_{z,Aq}(x)\right]^2,\end{aligned}$$

so that $Aq = z$. Since A and S are weak compatible of type (A) and $Aq = Sq = z$, $SAq = ASq$ and hence $Sz = SAq = ASq = Az$. Therefore, z is a common fixed point of A, B, S and T. Similarly, we can also complete the proof when B or S and T.

It follows easily from (4.4) that z is a unique common fixed point of A, B, S and T. This completes the proof.

As a consequence of Theorems 2.3 and 4.2, we have the following:

THEOREM 4.3. *Let A, B, S and T be mappings from a complete metric space (X, d) into itself such that*
(4.6) $A(X) \subset T(X)$ *and* $B(X) \subset S(X)$,
(4.7) *one of A, B, S and T is continuous,*
(4.8) *the pairs A, S and B, T are weak compatible of type (A),*

$$\begin{aligned} d^2(Ax, By) \le k \max\{ & d^2(Sx, Ty), d(Sx, Ax)\cdot d(Ty, By), \\ & d(Sx, Ty)\cdot d(Sx, Ax), d(Sx, Ty)\cdot d(Ty, By), \\ & \frac{1}{2} d(Sx, Ty)\cdot d(Sx, By), d(Sx, Ty)\cdot d(Ty, Ax), \\ & \frac{1}{2} d(Sx, By)\cdot d(Ty, Ax), d(Sx, Ax)\cdot d(Ty, Ax), \\ & \frac{1}{2} d(Sx, By)\cdot d(Ty, By)\}, \end{aligned} \tag{4.9}$$

for all x, y in X, where $k \in (0, 1)$.
Then A, B, S and T have a unique common fixed point in X.

REMARK 4. Theorems 4.2 and 4.3 generalize and improve the results of Cho-Murthy-Stojaković [6].

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