Comm. Korean Math. Soc. 10 (1995), No. 1, pp. 35–38

INTEGRABLE MODULES OVER QUANTUM GROUPS AT ROOTS OF 1

YOUNG HYUN CHO[1)], SAE RAN KWON AND IN-SOK LEE[2)]

1. Introduction

Let A be a symmetric positive definite Cartan matrix. As in [4], we denote by U the quantum group arising from A and U_λ be the corresponding quantum group at a root of unity λ. In [4], Lusztig constructed irreducible highest weight U_λ-modules $L_\lambda(z)$ for $z \in \mathbb{Z}^n$ and showed that $L_\lambda(z)$ is of finite dimension over $\mathbb{C}$ if and only if $z \in (\mathbb{Z}^+)^n$.

It is quite natural to ask what would be the integrable U_λ-modules. In this paper, we give a natural definition of integrable U_λ-modules and show that irreducible integrable highest weight U_λ-modules are also in one-to-one correspondence with $(\mathbb{Z}^+)^n$. Our result is based on [4, Theorem 7.4] which resembles the Steinberg's tensor product theorem (see Theorem 2.2).

As remarked in [3], the results of [3], [4] and hence the results of this paper can be obviously extended to an arbitrary positive definite Cartan matrix A (see Remark 2.6).

This paper can be considered as a supplement to Lusztig [4]. We will freely use the definitions and the notations given in [4] and will not repeat them here.

2. Integrable U_λ-modules

As in [4, 4.6], we only consider U_λ-modules of type **1**. If V is a U_λ-module of type **1** and $z \in \mathbb{Z}^n$, V_z denotes the z-weight space of V ([4, 5.2]).

Received November 4, 1994. Revised December 7, 1994.
Supported in part by SNU-Daewoo Program[1) 2)] and the GARC-KOSEF[2)].

DEFINITION 2.1. A U_λ-module V of type **1** is *integrable* if $V = \sum_{z\in\mathbb{Z}^n} V_z$ and if $E_i^{(l)}$ and $F_i^{(l)}$ $(i = 1,\dots,n)$ are locally nilpotent endomorphisms of V. We note that E_i and F_i $(i = 1,\dots,n)$ are always nilpotent on V. (Compare with [1, §3.6] and [3, 3.1].)

Let z be an arbitrary element of $\mathbb{Z}^n$. We can write uniquely $z = z' + lz''$ where $z' = (z'_1,\dots,z'_n)$, $z'' = (z''_1,\dots,z''_n) \in \mathbb{Z}^n$ and $0 \le z'_i \le l-1$ for all i. Using the Hopf algebra structure of U_λ, we can regard the tensor product $L_\lambda(z') \otimes L_\lambda(z'')$ as a U_λ-module.

THEOREM 2.2. ([4, Theorem 7.4]) *The U_λ-modules $L_\lambda(z)$ and $L_\lambda(z') \otimes L_\lambda(lz'')$ are isomorphic.*

Using above theorem we prove the following lemmas.

LEMMA 2.3. *If $E_i^{(l)}$ is locally nilpotent on $L_\lambda(z)$, then $E_i^{(l)}$ is also locally nilpotent on $L_\lambda(lz'')$.*

Proof. Let x' be a primitive vector of $L_\lambda(z')$ and v be any non-zero element of $L_\lambda(lz'')$. Then

$$E_i^{(l)}(x' \otimes v) = E_i^{(l)}(x') \otimes v + x' \otimes E_i^{(l)}(v) = x' \otimes E_i^{(l)}(v),$$

because x' is a primitive vector and so $E_i^{(l)}(x') = 0$. Since $L_\lambda(z) \cong L_\lambda(z') \otimes L_\lambda(lz'')$ and $E_i^{(l)}$ is locally nilpotent on $L_\lambda(z)$, $E_i^{(l)}$ is also locally nilpotent on $L_\lambda(z') \otimes L_\lambda(lz'')$. So there exist $m \in \mathbb{N}$ such that $(E_i^{(l)})^m(x' \otimes v) = 0$. But

$$(E_i^{(l)})^m(x' \otimes v) = x' \otimes (E_i^{(l)})^m(v).$$

So $(E_i^{(l)})^m(v) = 0$ and $E_i^{(l)}$ is locally nilpotent on $L_\lambda(lz'')$.

LEMMA 2.4. *If $F_i^{(l)}$ is locally nilpotent on $L_\lambda(z)$, then $F_i^{(l)}$ is also locally nilpotent on $L_\lambda(lz'')$.*

Proof. Let x' be a primitive vector of $L_\lambda(z')$ and v be any non-zero element of $L_\lambda(lz'')$. Then, since $z' \in (\mathbb{Z}^+)^n$, $\dim L_\lambda(z') < \infty$ by [4, Proposition 6.4]. Also by [4, Proposition 5.1], $F_i^{(l)}$ is nilpotent on $L_\lambda(z')$.

So there exist $m \in \mathbb{N}$ such that $(F_i^{(l)})^{m-1}.x' \neq 0$ and $(F_i^{(l)})^m.x' = 0$. Consider the element $(F_i^{(l)})^{m-1}(x') \otimes v \in L_\lambda(z') \otimes L_\lambda(z'')$. Then

$$\begin{aligned}&F_i^{(l)}.((F_i^{(l)})^{m-1}(x') \otimes v)\\ =&(F_i^{(l)})^m(x') \otimes v + (F_i^{(l)})^{m-1}(x') \otimes F_i^{(l)}(v)\\ =&(F_i^{(l)})^{m-1}(x') \otimes F_i^{(l)}(v).\end{aligned}$$

Therefore, for any $s \in \mathbb{N}$, we have

$$(F_i^{(l)})^s.((F_i^{(l)})^{m-1}(x') \otimes v) = (F_i^{(l)})^{m-1}(x') \otimes (F_i^{(l)})^s(v).$$

Also, since $F_i^{(l)}$ is locally nilpotent on $L_\lambda(z') \otimes L_\lambda(z'')$, there exist $n \in \mathbb{N}$ such that $(F_i^{(l)})^n((F_i^{(l)})^{m-1}(x') \otimes v) = 0$. Therefore

$$\begin{aligned}0 &= (F_i^{(l)})^n((F_i^{(l)})^{m-1}(x_0) \otimes v)\\ &= (F_i^{(l)})^{m-1}(x') \otimes (F_i^{(l)})^n(v).\end{aligned}$$

So $(F_i^{(l)})^n(v) = 0$ and $F_i^{(l)}$ is locally nilpotent on $L_\lambda(lz'')$.

Next, we return to the integrable U_λ-modules.

THEOREM 2.5. *The map $z \mapsto L_\lambda(z)$ induces a one-to-one correspondence between $(\mathbb{Z}^+)^n$ and the set of isomorphism classes of irreducible integrable highest weight U_λ-modules.*

Proof. If $z \in (\mathbb{Z}^+)^n$, then $L_\lambda(z)$ is clearly integrable by [4, Propositions 5.1 and 6.4]. Conversely, let $L_\lambda(z)$ be an irreducible integrable highest weight U_λ-module with highest weight $z \in \mathbb{Z}^n$. We can write uniquely $z = z' + lz''$ where $z', z'' \in \mathbb{Z}^n$, $z' = (z'_1, \dots, z'_n)$ and $0 \leq z'_i \leq l-1$ for all $i = 1, \dots, n$. By [4, Proposition 7.5], $L_\lambda(lz'')$ can be regarded as a $\overline{U}_1$-module and $L_\lambda(lz'')$ is an irreducible $\overline{U}_1$-module with highest weight z''. Now $E_i^{(l)}$ and $F_i^{(l)}$ are also locally nilpotent on $L_\lambda(lz'')$ by Lemma 2.3 and Lemma 2.4. Since $E_i \in \overline{U}_1$ acts as $E_i^{(l)} \in U_\lambda$ on $L_\lambda(lz'')$, $E_i \in \overline{U}_1$ are also locally nilpotent endomorphisms of $L_\lambda(lz'')$. Similary, $F_i \in \overline{U}_1$ are also locally nilpotent endomorphisms of $L_\lambda(lz'')$. Recall that $\overline{U}_1$ is the universal enveloping algebra of the simple Lie algebra corresponding to the Cartan matrix A. Hence $L_\lambda(lz'')$ is an irreducible integrable highest weight module with highest weight z'' over this simple Lie algebra. So by [1, Lemma 10.1], $z'' \in (\mathbb{Z}^+)^n$. Thus $z = z' + lz'' \in (\mathbb{Z}^+)^n$.

REMARK 2.6. The results in [3] and [4] are valid if A is an arbitrary positive definite Cartan matrix. (See [3, 4.14] and [2].) Therefore our results are also valid in that case.

References

1. V. G. Kac, *Infinite dimensional Lie algebras, 3rd ed.*, Cambridge Univ. Press, 1990.
2. M. Kashiwara, *On crystal bases of the q-analogue of universal enveloping algebras*, Duke Math. J. **63** (1991), 465-516.
3. G. Lusztig, *Quantum deformations of certain simple modules over enveloping algebras*, Adv. in Math. **70** (1988), 237-249.
4. G. Lusztig, *Modular representations and quantum groups*, Contemp. Math. **82** (1989), 59-77.

Young Hyun Cho
Department of Mathematics
Seoul National University
Seoul 151-742, Korea

Sae Ran Kwon
Daelim Junior College
Anyang 430-715, Korea

In-Sok Lee
Department of Mathematics
Seoul National University
Seoul 151-742, Korea