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UPPER BOUNDS FOR SUBPERMANENTS OF NONNEGATIVE MATRICES

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1. Introduction

For an $n \times n$ matrix $A = [a_{ij}]$, the *permanent* of A, per A, is defined by

$$\mathrm{per}(A) = \sum_{\sigma} a_{1\sigma(1)} \cdots a_{n\sigma(n)},$$

where σ runs over all permutations of $\{1, \cdots, n\}$.

For positive integers k, n such that $1 \leq k \leq n$, let $Q_{k,n}$ denote the set of all strictly increasing integer sequences of length k chosen from $1, \cdots, n$. For $\alpha, \beta \in Q_{k,n}$, and for an $n \times n$ matrix A, let $A[\alpha|\beta]$ denote the $k \times k$ submatrix of A lying in rows α and columns β. The permanent per$(A[\alpha|\beta])$ is called a *permanental k-minor* of A, or sometimes a *permanental minor* of A. The sum of all permanental k-minors of A is denoted by $\sigma_k(A)$. *i.e.*

$$\sigma_k(A) = \sum_{\alpha,\beta \in Q_{k,n}} \mathrm{per}\,(A[\alpha|\beta]). \tag{1.1}$$

We define $\sigma_0(A) = 0$, and note that $\sigma_n(A)$=per(A). We call $\sigma_k(A)$ a *k-th subpermanent* of A or *subpermanent* of A. If the matrix A is a (0,1)-matrix, then $\sigma_k(A)$ counts the number of different selections of k 1's of A with no two of the 1's on the same row or column.

A nonnegative square matrix whose row sums do not exceed one is called *substochastic*. We denote by $\hat{S}_n$ the set of all $n \times n$ substochastic matrices. Brualdi and Newman [1] obtained an upper bound for σ_k on $\hat{S}_n$.

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THEOREM 1.1. (Brualdi and Newman) *If $A \in \hat{S}_n$, then*

$$\sigma_k(A) \leq \binom{n}{k}, \qquad k = 1, \cdots, n \tag{1.2}$$

with equality holds for $k \geq 2$ if and only if A is a permutation matrix.

For general complex matrices, Marcus and Gordon [4] obtained upper bounds for the sum of the squares of absolute values of all permanental k-minors.

In this paper, we obtain an upper bound for the subpermanents on classes of general nonnegative matrices using the techniques of vector majorization, and we find the class of $n \times n$ substochastic matrices such that our bound for σ_k is shaper than that in (1.2).

2. Upper bounds for subpermanents

For integers k, n such that $1 \leq k \leq n$, let $V_{k,n}$ denote the set of all $n \times 1$ (0,1)-matrices whose entries have sum k. For real n-vectors, *i.e.*, real $n \times 1$ matrices $\mathbf{x}$ and $\mathbf{y}$, we say that $\mathbf{x}$ is *majorized* by $\mathbf{y}$ (or $\mathbf{y}$ majorizes $\mathbf{x}$),written as $\mathbf{x} \prec \mathbf{y}$ if

$$\max\{\mathbf{v}^{\mathrm{T}}\mathbf{x} | \mathbf{v} \in V_{k,n}\} \ \leq \ \max\{\mathbf{v}^{\mathrm{T}}\mathbf{y} | \mathbf{v} \in V_{k,n}\} \tag{2.1}$$

for all $k = 1, \cdots, n$ and equality holds in (2.1) when $k = n$. $\mathbf{x}$ is said to be *submajorized* by $\mathbf{y}$, written as $\mathbf{x} \prec_{\bullet} \mathbf{y}$ if (2.1) holds for all $k = 1, \cdots, n$. Similarly $\mathbf{x}$ is said to be *supermajorized* by $\mathbf{y}$, written as $\mathbf{x} \prec^{\bullet} \mathbf{y}$ if

$$\min\{\mathrm{v}^{\mathrm{T}}\mathbf{x} | \mathbf{v} \in V_{k,n}\} \ \geq \ \min\{\mathbf{v}^{\mathrm{T}}\mathbf{y} | \mathbf{v} \in V_{k,n}\}. \tag{2.2}$$

EXAMPLE 2.1. For an $n \times n$ real matrix $A = [a_{ij}]$ with per $A \neq 0$, let $S = [s_{ij}]_{n \times n}$ where $s_{ij} = a_{ij}\text{per } A(i|j)/\text{per } A$, $i, j = 1, \cdots, n$. Then

$$\left(\frac{1}{n}, \cdots, \frac{1}{n}\right)^{\mathrm{T}} \prec (s_{i1}, \cdots, s_{in})^{\mathrm{T}}, \quad i = 1, \cdots, n.$$

From now on, for $k \in \{1, \cdots, n\}$, let S_k denote the k-th elementary symmetric function of $\mathbf{R}^n$, *i.e.*,

$$\mathrm{S}_k(\mathbf{x}) = \sum_{\alpha \in Q_{k,n}} \prod_{i \in \alpha} x_i \tag{2.3}$$

for $\mathbf{x} = (x_1, \cdots, x_n)^{\mathrm{T}} \in \mathbf{R}^n$.

The following lemma provides a very useful mechanism for determining some upper bounds for subpermanents.

LEMMA 2.2. (Muirhead's Theorem [7]) *Let* $\mathbf{x} = (x_1, x_2, \cdots, x_n)^{\mathrm{T}} \in \mathbf{R}^n$ *and* $\mathbf{y} = (y_1, y_2, \cdots, y_n)^{\mathrm{T}} \in \mathbf{R}^n$. *Then for all positive real n-vectors* $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)^{\mathrm{T}}$, $\mathbf{x} \prec \mathbf{y}$ *if and only if*

$$\sum_{\sigma} \alpha_{\sigma(1)}^{x_1} \alpha_{\sigma(2)}^{x_2} \cdots \alpha_{\sigma(n)}^{x_n} \leq \sum_{\sigma} \alpha_{\sigma(1)}^{y_1} \alpha_{\sigma(2)}^{y_2} \cdots \alpha_{\sigma(n)}^{y_n}, \tag{2.4}$$

where σ runs over all permutations of $\{1, \cdots, n\}$. *Similarly,*

$$\mathbf{x} \prec_{\bullet} \mathbf{y} \ \textit{if and only if} \ (2.4) \textit{ holds for all } \alpha \in [1, \infty)^n. \tag{2.5}$$

$$\mathbf{x} \prec^{\bullet} \mathbf{y} \ \textit{if and only if} \ (2.4) \textit{ holds for all } \alpha \in (0, 1]^n. \tag{2.6}$$

THEOREM 2.3. *If $A = [a_{ij}]$ is an $n \times n$ nonnegative matrix then*

$$\sigma_k(A) \leq \binom{n}{k}^2 \frac{k!}{n^2} \sum_{i,j} a_{ij}^k, \quad k = 1, \cdots, n, \tag{2.7}$$

with equality holds in (2.7) if A is a scalar multiple of $J_n = [1/n]_{n \times n}$ or $k = 1$.

In particular,

$$\mathrm{per}(A) \leq \frac{n!}{n^2} \sum_{i,j} a_{ij}^n. \tag{2.8}$$

Proof. We prove (2.7) for an $n \times n$ positive matrix $A = [a_{ij}]$ first. Let $\mathcal{S}_n$ denote the symmetric group on the set $\{1, \cdots, n\}$. For $\sigma \in \mathcal{S}_n$, let $\mathbf{d}_\sigma = (d_1, \cdots, d_n) := (a_{1\sigma(1)}, \cdots, a_{n\sigma(n)})$. Then since

$$(\underbrace{1, \cdots, 1}_{k\ times}, 0, \cdots, 0)^{\mathrm{T}} \prec (k, 0, \cdots, 0)^{\mathrm{T}},$$

we have from (2.4), that

$$\sum_{\tau \in \mathcal{S}_n} d_{\tau(1)} \cdots d_{\tau(k)} \le \sum_{\tau \in \mathcal{S}_n} d_{\tau(1)}^{\,k}. \tag{2.9}$$

Note that

$$\sum_{\tau \in \mathcal{S}_n} d_{\tau(1)} \cdots d_{\tau(k)} = k!(n-k)!\mathrm{S}_k(\mathbf{d}_\sigma) \tag{2.10}$$

and

$$\sum_{\tau \in \mathcal{S}_n} d_{\tau(1)}^{\,k} = (n-1)!(d_1^k + \cdots + d_n^k). \tag{2.11}$$

Summing (2.10) and (2.11) respectively over $\sigma \in \mathcal{S}_n$ we have

$$k!(n-k)! \sum_{\sigma \in \mathcal{S}_n} \mathrm{S}_k(\mathbf{d}_\sigma) \le \{(n-1)!\}^2 \sum_{i,j=1}^{n} a_{ij}^k. \tag{2.12}$$

Since

$$\sum_{\sigma \in \mathcal{S}_n} \mathrm{S}_k(\mathbf{d}_\sigma) = (n-k)!\sigma_k(A), \tag{2.13}$$

we have

$$k!\{(n-k)!\}^2 \sigma_k(A) \le \{(n-1)!\}^2 \sum_{i,j} a_{ij}^k$$

which is equivalent to the required inequality

$$\sigma_k(A) \le \binom{n}{k}^2 \frac{k!}{n^2} \sum_{i,j} a_{ij}^k.$$

Once the inequality (2.7) holds for positive matrices, then we can show that it holds for nonnegative matrices too using the limiting process.

If $k = n$ in (2.7) then we obtain the inequality (2.8), and it is easy to show that the equality in (2.7) holds if A is a scalar multiple of $J_n = [1/n]_{n\times n}$ or $k = 1$. This completes the proof.

Now, we will show that the upper bound of $\sigma_k(A)$ in (2.7) for A belongs to a certain class of $n \times n$ substochastic matrices is shaper than that in (1.2). We shall require the following Lemma.

LEMMA 2.4. *For positive integers k, n such that $2 \leq k \leq n$,*

$$\left(\frac{1}{\sqrt{n(n-1)}}\right)^k \leq \frac{(n-k)!}{n!} \tag{2.14}$$

with equality holds for $k = 2$.

Proof. We use induction on k for a fixed n. If $k = 2$ then the equality holds. If $k = 3$ then it is easy to show that (2.14) holds. Now assume that (2.14) holds for $k = m$. Then we get

$$\begin{aligned}\left(\frac{1}{\sqrt{n(n-1)}}\right)^{m+1} &\leq \frac{(n-m-1)!}{n!}\left(\frac{(n-m)(n-m)}{n(n-1)}\right)^{\frac{1}{2}}\\ &\leq \frac{(n-m-1)!}{n!},\end{aligned}$$

which completes the proof.

THEOREM 2.5. *Let $A = [a_{ij}]$ be an $n \times n$ substochastic matrix such that*

$$\max_{i,j}\{a_{ij}\} \leq \frac{1}{\sqrt{n(n-1)}}. \tag{2.15}$$

Then the upper bound of $\sigma_k(A)$ in (2.7) improves that of Brualdi and Newman in (1.2).

Proof. It suffices to show that

$$\binom{n}{k}^2 \frac{k!}{n^2}\sum_{i,j} a_{ij}^k \leq \binom{n}{k}, \quad k = 1, \cdots, n \tag{2.16}$$

for any $n \times n$ substochastic matrix $A = [a_{ij}]$ with (2.15).

Note that if $k = 1$ then (2.16) holds. Now, let $k \geq 2$. Then from (2.15) and (2.14) we get

$$\begin{aligned}\binom{n}{k}^2 \frac{k!}{n^2} \sum_{i,j} a_{ij}^k &\leq \binom{n}{k}^2 k! \left(\frac{1}{\sqrt{n(n-1)}}\right)^k \\ &\leq \binom{n}{k}^2 \frac{k!(n-k)!}{n!} \\ &= \binom{n}{k},\end{aligned}$$

which completes the proof.

EXAMPLE 2.6. Let A be 3×3 substochastic matrix of the form

$$A = \frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ 1 & 2 & 4 \\ 2 & 1 & 2 \end{bmatrix}.$$

Then we get from the bounds given by Theorem 2.3 and (1.2) the following:

k	$\sigma_k(A)$	$\binom{n}{k}^2 \frac{k!}{n^2} \sum_{ij} a_{ij}^k$	$\binom{n}{k}$
1	2	2	3
2	$\frac{89}{100}$	$\frac{108}{100}$	3
3	$\frac{62}{1000}$	$\frac{113}{1000}$	1

A conjecture related to the subpermanent is the following one proposed by Doković [2] :

CONJECTURE I. (Doković) *If A is an $n \times n$ doubly stochastic matrix then*

$$\frac{(n-k+1)^2}{nk}\sigma_{k-1}(A) \leq \sigma_k(A), \quad k = 1, \cdots, n \tag{2.17}$$

The Doković conjecture for $k \leq 3$ was proved by Doković himself [2].

THEOREM 2.7. *Let $A = [a_{ij}]$ be an $n \times n$ nonnegative matrix. Then*

$$(2.18) \qquad (a) \quad \sigma_k(A) \le \frac{(n-k+1)^2}{k}\sigma_{k-1}(A), \quad k = 1, \cdots, n,$$

if $0 \le a_{ij} \le 1$ *for all* $i, j = 1, \cdots, n$, and

$$(2.19) \qquad (b) \quad \sigma_k(A) \ge \frac{(n-k+1)^2}{k}\sigma_{k-1}(A), \quad k = 1, \cdots, n,$$

if $a_{ij} \ge 1$ *for all* $i, j = 1, \cdots, n$.

Proof. (a) The limiting process enables us to assume that $0 < a_{ij} \le 1$ for all $i, j = 1, \cdots, n$. Since

$$(\underbrace{1, \cdots, 1}_{k\ times}, 0, \cdots, 0) \prec^{\bullet} (\underbrace{1, \cdots, 1}_{(k-1)\ times}, 0, \cdots, 0),$$

we have from (2.6),(2.10) and (2.13), that

$$k!\{(n-k)!\}^2\sigma_k(A) \le (k-1)!\{(n-k+1)!\}^2\sigma_{k-1}(A)$$

which is equivalent to the inequality (2.18).
(b) The inequality (2.19) can be proved similarly. This completes the proof.

REMARK. If we confirm the inequality (2.17), then from (2.18) we have

$$\frac{(n-k+1)^2}{nk}\sigma_{k-1}(A) \le \sigma_k(A) \le \frac{(n-k+1)^2}{k}\sigma_{k-1}(A), \quad k = 1, \cdots, n$$

for every doubly stochastic matrix A.

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