J. Korean Math. Soc. **32** (1995), No. 4, pp. 849–856

SPANNING COLUMN RANK 1 SPACES OF NONNEGATIVE MATRICES

SEOK-ZUN SONG, GI-SANG CHEON AND GWANG-YEON LEE

1. Introduction

There are some papers on structure theorems for the spaces of matrices over certain semirings. Beasley, Gregory and Pullman [1] obtained characterizations of semiring rank 1 matrices over certain semirings of the nonnegative reals. Beasley and Pullman [2] also obtained the structure theorems of Boolean rank 1 spaces. Since the semiring rank of a matrix differs from the column rank of it in general, we consider a structure theorem for semiring rank in [1] in view of column rank.

In this paper, we obtain a characterization of column rank 1 matrices and a structure theorem for the vector space of matrices whose nonzero members all have spanning column rank 1 over nonnegative part of a unique factorization domain that is not a field in the reals.

2. Definitions and preliminaries

Let $\mathbf{R}$ denote the field of reals and $\mathbf{S}$ denote an arbitrary semiring of nonnegative reals. Let $\mathbf{U}_+$ be the nonnegative part of a unique factorization domain which is not a field in $\mathbf{R}$. Such examples are $\mathbf{Z}_+, (\mathbf{Q}[\pi])_+$ etc., where $\mathbf{Z}, \mathbf{Q}$ denote the rings of integers and rationals, respectively, and π is a transcendental number over $\mathbf{Q}$.

Let $\mathbf{A}$ be an $m \times n$ matrix over $\mathbf{S}$. If $\mathbf{A}$ is a nonzero matrix, then the *semiring rank* [3] of $\mathbf{A}, r(\mathbf{A})$, is the least k for which there exist $m \times k$ and $k \times n$ matrices F and G over $\mathbf{S}$ such that $A = FG$. The zero matrix is assigned the semiring rank 0. The set of $m \times n$ matrices

Received November 2, 1994.

1991 AMS Subject Classification: Primary 15A03, 15A23.

Key words: Spanning column rank, spanning column rank 1 space.

Supported by Korea Science and Engineering Foundation 941-0100-027-2, and in part by TGRC-KOSEF, 1994.

with entries in $\mathbf{S}$ is denoted by $\mathbb{M}_{m,n}(\mathbf{S})$. Addition, multiplication by scalars, and the product of matrices are defined as if $\mathbf{S}$ were a field.

If $\mathbb{V}$ is a nonempty subset of $\mathbf{S}^k \equiv \mathbb{M}_{k,1}(\mathbf{S})$ that is closed under addition and multiplication by scalars, then $\mathbb{V}$ is called a *vector space* over $\mathbf{S}$. The notions of subspace and of spanning sets are the same as if $\mathbf{S}$ were a field. As with fields, a *basis* for a vector space $\mathbb{V}$ is a spanning subset of least cardinality. That cardinality is the *dimension*, $\dim(\mathbb{V})$, of $\mathbb{V}$.

For an $m \times n$ matrix A over $\mathbf{S}$, the *column rank* [5], $c(A)$, is the dimension of the vector space spanned by its columns, and the *spanning column rank* [4], $sc(A)$, is the minimum number of the columns of A which span its column space.

It follows that

$$0 \le r(A) \le c(A) \le sc(A) \le n \tag{2.1}$$

for all $m \times n$ matrices A over $\mathbf{S}$. But these rank functions may differ over certain semirings as shown in the following example.

EXAMPLE 2.1. Consider a matrix $A = [3, 6 - 2\sqrt{7}, 2\sqrt{7} - 4]$ over a semiring $\mathbf{S} = (\mathbf{Z}[\sqrt{7}])_+$. Then it is trivially that $r(A) = 1$. Since $(6 - 2\sqrt{7}) + (2\sqrt{7} - 4) = 2$, 2 is spanned by the last two columns of A. Then we have $(6 - 2\sqrt{7}) = 2(3 - \sqrt{7})$ and $2\sqrt{7} - 4 = 2(\sqrt{7} - 2)$ with $3 - \sqrt{7}, \sqrt{7} - 2 \in \mathbf{S}$, which means that $\{2, 3\}$ is a basis of the column space of A. So $c(A) = 2$. But, any column of A cannot be spanned by the other two columns. That is, $sc(A) = 3$. ■

Let Γ be a nonempty subset of $\mathbf{S}^k$ and let $\mathbf{g} \in \mathbf{S}^k$. We'll say that $\mathbf{g}$ is a *common factor* of Γ if $\Gamma \subseteq \{\sigma\mathbf{g} \mid \sigma \in \mathbf{S}\}$.

LEMMA 2.2. ([1]) *Let Γ be any nonempty subset of $(\mathbf{U}_+)^k$. Each pair of nonzero vectors in Γ has a common nonzero scalar multiple in $(\mathbf{U}_+)^k$ if and only if Γ has a common factor in $(\mathbf{U}_+)^k$.* ■

EXAMPLE 2.3. If $k > 1$, let

$$A(k) = \begin{pmatrix} 1 & 1 & k-1 \\ 1 & k & 0 \\ 1 & 0 & k \end{pmatrix}.$$

If $0 < k < 1$, let $p = [\frac{1}{k}]$, $q = p - 1$ and

$$A(k) = \begin{pmatrix} 1 & 1-kq & kp-1 \\ 1 & k & 0 \\ 1 & 0 & k \end{pmatrix}.$$

If k is a nonzero nonunit in $\mathbf{S}$, then $c(A(k)) = 3$ by definition of column rank. Multiplying the first column of $A(k)$ by k reduces its column rank to 2. From this matrix $A(k)$ we can obtain an $m \times n$ matrix of column rank r such that the matrix obtained by multiplying the jth column of it by k has column rank $r-1$ as follows ; let P be the matrix obtained from I_n by interchanging I_n's first and jth column, and let B be any $(m-3) \times (n-3)$ matrix over $\mathbf{S}$ of column rank $r-3$. Then $X = (A \oplus B)P$ is the required matrix of column rank r. ■

3. Column rank 1 matrix

If X is a matrix over a semiring $\mathbf{S}$ and $X = \mathbf{x}\mathbf{a}^t$, then the vectors $\mathbf{x}, \mathbf{a}$ are called *left* and *right factors* of X respectively. In particular, $\mathbf{a}$ is called a *basic right factor* of X if $\mathbf{a}^t$ has column rank 1.

THEOREM 3.1. *For $A \in \mathbb{M}_{m,n}(\mathbf{S}), c(A) = 1$ if and only if A can be factored as $\mathbf{x}\mathbf{a}^t$ for some $\mathbf{a} \in \mathbf{S}^n, \mathbf{x} \in \mathbf{S}^m$, where $\mathbf{x} \neq 0$ and $\mathbf{a}^t$ is a basic right factor.*

Proof. Suppose that $c(A) = 1$ and denote the columns of A by $\mathbf{a}_1, \cdots, \mathbf{a}_n$. Let $\{\mathbf{x}\}$ be a basis of the column space of A over $\mathbf{S}$, so that $\mathbf{x} = \sum_{j=1}^n \gamma_j \mathbf{a}_j$ for some constants $\gamma_1, \cdots, \gamma_n$ in $\mathbf{S}$. In particular, $\mathbf{x} \in \mathbf{S}^m$ and $\mathbf{x} \neq \mathbf{0}$. Now for each j between 1 and n, we have $\mathbf{a}_j = \alpha_j \mathbf{x}$ for some $\alpha_j \in \mathbf{S}$, since $\mathbf{x}$ spans the column space of A. Letting $\mathbf{a}^t = [\alpha_1, \cdots, \alpha_n]$, we have $\mathbf{a} \in \mathbf{S}^n$ and $A = \mathbf{x}\mathbf{a}^t$. Further, $\mathbf{x} = \sum_{j=1}^n \gamma_j \mathbf{a}_j = \sum_{j=1}^n \gamma_j \alpha_j \mathbf{x}$, and hence $1 = \sum_{j=1}^n \gamma_j \alpha_j$ since $\mathbf{x}$ is not zero. Thus 1 is in the column space of $\mathbf{a}^t$, and it follows that $c(\mathbf{a}^t) = 1$. Consequently, $\mathbf{a}$ is a basic right factor of A, as desired.

The converse is clear. ■

Identifying $\mathbf{S}^{mn}$ with $\mathbb{M}_{m,n}(\mathbf{S})$, we transfer the definitions to $\mathbb{M}_{m,n}(\mathbf{S})$. If $\mathbb{V} \neq \{0\}$ is a vector space in $\mathbb{M}_{m,n}(\mathbf{S})$ whose members have column rank at most 1, then $\mathbb{V}$ is a *column rank* 1 *space*. If $\mathbb{V}$ is a vector space all of whose members have the same basic right factor $\mathbf{b}$, then $\mathbb{V}$ is called a *basic right factor space*. Notice that in that case $\mathbb{W} = \{\mathbf{a} \in \mathbf{S}^m \mid \mathbf{ab}^t \in \mathbb{V}\}$ is a vector space in $\mathbf{S}^m$. Conversely, if $\mathbb{W}$ is a vector space in $\mathbf{S}^m$ and $c(\mathbf{b}^t) = 1$ then $\{\mathbf{ab}^t \mid \mathbf{a} \in \mathbb{W}\}$ is a basic right factor space in $\mathbb{M}_{m,n}(\mathbf{S})$. Evidently basic right factor spaces are column rank 1 spaces.

Define a relation λ on the $m \times n$ column rank 1 matrices over $\mathbf{S}$ by :$A\lambda B$ if A and B have a common basic right factor.

PROPOSITION 3.2. (1) *λ is an equivalence relation on the $m \times n$ column rank 1 matrices over $\mathbf{U}_+$.*
(2) *For any nonempty set E of $m \times n$ column rank 1 matrices over $\mathbf{U}_+$, the members of E have a common basic right factor if and only if $X\lambda Y$ for all X, Y in E.*

Proof. (1) Evidently λ is reflexive and symmetric. Suppose A, B, C are $m \times n$ column rank 1 matrices over $\mathbf{U}_+$ that satisfy $A\lambda B$ and $B\lambda C$. Then A, B and C can be factored as $A = \mathbf{xa}^t, \mathbf{ya}^t = B = \mathbf{zb}^t$ and $C = \mathbf{wb}^t$ by Theorem 3.1, where $\mathbf{a}^t$ and $\mathbf{b}^t$ have column rank 1. Now $\mathbf{a}, \mathbf{b}$ have a common nonzero scalar multiple because the left factors of B are nonzero. Therefore $\mathbf{a}, \mathbf{b}$ have a common factor $\mathbf{f}$ by Lemma 2.2. and $\mathbf{f}^t$ has column rank 1. So A and C can be factored as $A = (\alpha\mathbf{x})\mathbf{f}^t$ and $C = (\beta\mathbf{w})\mathbf{f}^t$ for some $\alpha, \beta \in \mathbf{U}_+$. Consequently $A\lambda C$ and hence λ is transitive.

(2) Suppose $X\lambda Y$ for all X, Y in E. For each X in E, select a basic right factor $\mathbf{g}_X$ and put $\Gamma = \{\mathbf{g}_X \mid X \in E\}$. By the proof of (1), if A, B are in E, then A and B have a common basic right factor. Thus $\mathbf{g}_A$ and $\mathbf{g}_B$ have a common nonzero scalar multiple. Therefore Γ has a common factor $\mathbf{f}$ by Lemma 2.2, and $\mathbf{f}^t$ has column rank 1. Thus $\mathbf{f}$ is a common basic right factor of all X in E.

The converse is immediate. ■

Thus the λ-equivalence classes are the maximal basic right factor spaces in $\mathbb{M}_{m,n}(\mathbf{U}_+)$. These in turn are of the form $V(\mathbf{a}) = \{\mathbf{xa}^t \mid \mathbf{x} \in \mathbf{U}_+^m\}$, where $c(\mathbf{a}^t) = 1$.

4. Spanning column rank 1 spaces

In this section, we obtain a structure theorem for the vector space of matrices whose members have spanning column rank at most 1. For this purpose we need some definitions and lemmas.

If A is a matrix over a semiring $\mathbf{S}$ and A has the form $\mathbf{fa}^t$, then $\mathbf{a}$ is called a *strong right factor* of A if $\mathbf{a}^t$ has spanning column rank 1. Hwang, Kim and Song [4] showed the following Lemma:

LEMMA 4.1. ([4]) *For $A \in \mathbb{M}_{m,n}(\mathbf{S}), sc(A) = 1$ if and only if A can be factored as $\mathbf{fa}^t$ for some $\mathbf{a} \in \mathbf{S}^n$ and $\mathbf{f} \in \mathbf{S}^m$, where $\mathbf{f} \neq \mathbf{0}$ and $\mathbf{a}^t$ is a strong right factor.* ∎

If $V \neq \{\mathbf{0}\}$ is a vector space in $\mathbb{M}_{m,n}(\mathbf{S})$ whose members have spanning column rank at most 1, then V is called a *spanning column rank* 1 *space.* If V is a vector space all of whose members have the same strong right factor $\mathbf{b}$, then V is called a *strong right factor space.* As the case of basic right factor space, $W = \{\mathbf{a} \in \mathbf{S}^m \mid \mathbf{ab}^t \in V\}$ is a vector space in $\mathbf{S}^m$. Conversely, if W is a vector space in $\mathbf{S}^m$ and $sc(\mathbf{b}^t) = 1$ then $\{\mathbf{ab}^t \mid \mathbf{a} \in W\}$ is a strong right factor space in $\mathbb{M}_{m,n}(\mathbf{S})$. Evidently strong right factor spaces are spanning column rank 1 spaces.

Beasley and Pullman [1] obtained a Lemma for the common factor of two matrices as follows:

LEMMA 4.2. ([1]) *Suppose A and B are $m \times n$ matrices of semiring rank 1 over $\mathbf{U}_+$ and $\min(m, n) \geq 2$. Then $r(A + B) = 1$ if and only if A and B have a common factor.* ∎

For the common strong right factor of two matrices, we obtain the following Lemma :

LEMMA 4.3. *Suppose $A, B \in \mathbb{M}_{m,n}(\mathbf{U}_+)$ with $sc(A) = sc(B) = 1$ and $\min(m, n) \geq 2$. Then A and B have a common strong right factor if and only if $sc(\alpha A + \beta B) = 1$ for any $\alpha, \beta \in \mathbf{U}_+$, not both zero.*

Proof. By Lemma 4.1, we can write $A = \mathbf{fa}^t$, and $B = \mathbf{gb}^t$ for some $\mathbf{f}, \mathbf{g} \in (\mathbf{U}_+)^m$ and $\mathbf{a}, \mathbf{b} \in (\mathbf{U}_+)^n$ with $sc(\mathbf{a}^t) = sc(\mathbf{b}^t) = 1$. Assume that A and B have a common strong right factor $\mathbf{r}$. Then, for any $\alpha, \beta \in \mathbf{U}_+, \alpha A + \beta B = (\alpha\sigma\mathbf{f} + \beta\tau\mathbf{g})\mathbf{r}^t$ for some $\sigma, \tau \in \mathbf{U}_+$. Since $sc(\mathbf{r}^t) = sc(\sigma\mathbf{r}^t) = sc(\mathbf{a}^t) = 1, sc(\alpha A + \beta B) = 1$ for any α, β, not both zero.

Conversely, assume that $sc(\alpha A + \beta B) = 1$ for any $\alpha, \beta \in \mathbf{U}_+$, not both zero. Then we have $r(\alpha A + \beta B) = 1$ by (2.1). In particular, A and B have a common factor by Lemma 4.2.

Case 1) A and B have a common right factor $\mathbf{r}$. Then we can write $A + B = (\sigma\mathbf{f} + \tau\mathbf{g})\mathbf{r}^t$ for some $\sigma, \tau \in \mathbf{U}_+$. Since $sc(\mathbf{r}^t) = sc(\sigma\mathbf{r}^t) = sc(\mathbf{a}^t) = 1, A$ and B have a common strong right factor $\mathbf{r}$.

Case 2) A and B have a common left factor $\mathbf{d}$. Then we may write $A = \mathbf{d}\alpha\mathbf{a}^t$ and $B = \mathbf{d}\beta\mathbf{b}^t$, where $\alpha\mathbf{a} = (a_1, \cdots, a_n)^t$, and $\beta\mathbf{b} = (b_1, \cdots, b_n)^t$ are strong right factors of A and B, respectively. Since there are infinitely many primes in $\mathbf{U}_+$ (for the existence of infinite primes, see Lemma 2.2 in [4]), we can choose a prime π such that π does not divide all nonzero $b_i, i = 1, \cdots, n$. Consider

$$\pi^p A + B = \mathbf{d}[\pi^p a_1 + b_1, \pi^p a_2 + b_2, \cdots, \pi^p a_n + b_n]$$

which has spanning column rank 1 for any positive integer p. Since the columns of $\pi^p A + B$ are finite in number, there exists a column j and a sequence of p's with the properties that i) the jth columns of $\pi^p A + B$ spans the column space for each term p in the sequence, and ii) the difference between two successive terms in the sequence is at most n. Therefore for infinitely many p,

$$\pi^p a_k + b_k = \mu_{pk}(\pi^p a_j + b_j) \tag{4.1}$$

for some $\mu_{hk} \in \mathbf{U}_+, k = 1, \cdots, n$. In (4.1), if $b_j = 0$, then b_k must be divided by nonunit π^p. But it is impossible since π does not divide b_k for at least one nonzero b_k. Thus $b_j \neq 0$. If the column space of $\pi^q A + B$ is spanned by its jth column, then we get

$$\pi^q a_k + b_k = \mu_{qk}(\pi^q a_j + b_j) \tag{4.2}$$

for some $\mu_{qk} \in \mathbf{U}_+, k = 1, \cdots, n$. From (4.1) and (4.2), we get $\mid \mu_{qk} - \mu_{pk} \mid \in \mathbf{U}_+$ for $q > p$. Since there are only n columns in $\pi^p A + B$ for each p, we can choose infinitely many pairs p and q such that they satisfy $p < q \leq p + n$ and the column spaces of $\pi^p A + B$ and $\pi^q A + B$ are spanned by their jth column respectively. For such pairs p and q, consider

$$\begin{aligned} \mid \mu_{qk} - \mu_{pk} \mid &= \left| \frac{\pi^q a_k + b_k}{\pi^q a_j + b_j} - \frac{\pi^p a_k + b_k}{\pi^p a_j + b_j} \right. \\ &= \frac{|(\pi^{q-p} - 1)(a_k b_j - a_j b_k)|\pi^p}{(\pi^q a_j + b_j)(\pi^p a_j + b_j)} \end{aligned} \tag{4.3}$$

Assume that $\mu_{qk} \neq \mu_{pk}$ for all such pairs p and q. Since π is prime, π is not divided by $\pi^p a_j + b_j$. If $\pi^p a_j + b_j$ has π as its prime factor, then $\pi^p a_j + b_j = \beta\pi$ for some $\beta \in \mathbf{U}_+$. Thus $\pi(\beta - \pi^{p-1}a_j) = b_j$ and hence b_j is divided by π, which is a contradiction. Then π^p does not have any factor of $(\pi^p a_j + b_j)(\pi^q a_j + b_j)$. Since $| a_k b_j - a_j b_k |$ is fixed and $| \pi^{q-p} - 1 |$ takes at most n values for any pairs p and q with $1 \leq q - p \leq n$, the prime factors of $| (\pi^{q-p} - 1)(a_k b_j - a_j b_k) |$ are finite in number. Thus we can choose sufficiently large pair p and q with $1 \leq q - p \leq n$ such that $| (\pi^{q-p} - 1)(a_k b_j - a_j b_k) |$ does not contain some prime factors of $(\pi^p a_j + b_j)(\pi^q a_j + b_j)$. Then the denominator of (4.3) contains some nonunit prime factors such that the numerator of (4.3) does not contain. Since $\mathbf{U}_+$ contains no element of the form $\frac{x}{y}$, where y has a prime factor which x does not, the fractional expression of (4.3) is not an element of $\mathbf{U}_+$. Thus we have a contradiction such that $| \pi_{qk} - \pi_{pk} | \notin \mathbf{U}_+$ for some pair p and q with $p < q \leq p + n$. Hence $\mu_{qk} = \mu_{pk}$ for some p and q. Subtracting (4.1) from (4.2), we have $a_k = \mu_{pk} a_j$ for all $k = 1, \cdots, n$. And we get $b_k = \mu_{pk} b_j$ for all $k = 1, \cdots, n$ from (4.1). That is, $\mathbf{a} = a_j \mathbf{r}$ and $\mathbf{b} = b_j \mathbf{r}$ where $\mathbf{r} = [\mu_{p1}, \cdots, \mu_{pn}]$ with $\mu_{pj} = 1$.

By cases 1) and 2), A and B have a common strong right factor $\mathbf{r}$. ■

Define a relation ρ on the $m \times n$ spanning column rank 1 matrices over a semiring $\mathbf{S}$ by : $A \rho B$ if A, B have a common strong right factor. Then we have some properties on the relation ρ that are similar to those on the relation λ in section 3.

PROPOSITION 4.4.

(1) *ρ is an equivalence relation on the $m \times n$ spanning column rank 1 matrices over $\mathbf{U}_+$.*

(2) *For any nonempty set F of $m \times n$ spanning column rank 1 matrices over $\mathbf{U}_+$, the members of F have a common strong right factor if and only if $X \rho Y$ for all X, Y in F.*

Proof. Similar to the proof of Proposition 3.2. ■

Thus the ρ-equivalence classes are the maximal strong right factor spaces in $\mathbb{M}_{m,n}(\mathbf{U}_+)$. These in turn are of the form $\mathbf{V}(\mathbf{a}) = \{\mathbf{x}\mathbf{a}^t \mid \mathbf{x} \in (\mathbf{U}_+)^m\}$, where $\mathbf{a}^t$ has spanning column rank 1.

THEOREM 4.5. *Suppose that V is a subspace of $\mathbb{M}_{m,n}(\mathbf{U}_+)$ with $\min(m,n) \geq 2$. Then V is a spanning column rank 1 space if and only if V is a strong right factor space.*

Proof. Suppose V is a spanning column rank 1 space. For every A and B in V, $sc(\alpha A + \beta B) = 1$ for any $\alpha, \beta \in \mathbf{U}_+$, not both zero. Then A and B have a common strong right factor by Lemma 4.3. Therefore V is a strong right factor space by Proposition 4.4.

The converse is immediate. ■

Thus we have a structure theorem for spanning column rank 1 space in $\mathbb{M}_{m,n}(\mathbf{U}_+)$.

References

1. L. B. Beasley, D. A. Gregory and N. J. Pullman, *Nonnegative rank-preserving operators*, Linear Algebra Appl. **65** (1985), 207-223.
2. L. B. Beasley and N. J. Pullman, *Boolean rank-preserving operators and Boolean rank-1 spaces*, Linear Algebra Appl. **59** (1984), 55-77.
3. L. B. Beasley and S. Z. Song, *A comparison of nonnegative real ranks and their preservers*, Linear and Multilinear Algebra **31** (1992), 37-46.
4. S. G. Hwang, S. J. Kim and S. Z. Song, *Linear operators that preserve spanning column rank of nonnegative matrices*, J. Korean Math. Soc. **31** (1994), 645-657.
5. S. Z. Song, *Linear operators that preserve column rank of Boolean matrices*, Proc. Amer. Math. Soc. **119** (1993), 1085-1088.
6. S. Z. Song, *Linear operators that preserve column rank of fuzzy matrices*, Fuzzy Sets and Systems **62** (1994), 311-317.

Seok-Zun Song
Department of Mathematics
Cheju National University
Cheju 690-756, Korea

Gi-Sang Cheon
Department of Mathematics
Dae Jin University
Pocheon 487-800, Korea

Gwang-Yeon Lee
Department of Mathematics
Hanseo University
Seosan, Chung-Nam 352-820, Korea