J. Korean Math. Soc. **32** (1995), No. 4, pp. 803–814

PLANAR HARMONIC MAPPINGS AND CURVATURE ESTIMATES

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1. Introduction

Let Σ be the class of all complex-valued, harmonic, orientation-preserving, univalent mappings defined on $\Delta = \{z : |z| > 1\}$ that map ∞ to ∞. Then $f \in \Sigma$ has the representation

$$f(z) = h(z) + \overline{g(z)} + A \log |z| \tag{1.1}$$

where

$$h(z) = \alpha z + \sum_{k=0}^{\infty} a_k z^{-k} \quad \text{and} \quad g(z) = \beta z + \sum_{k=1}^{\infty} b_k z^{-k}$$

are analytic in Δ and $0 \le |\beta| < |\alpha|$ (see [3]). In addition, f can be viewed as a solution of the partial differential equation

$$\overline{f_{\bar{z}}} = a f_z \tag{1.2}$$

where the function $a = \frac{\overline{f_{\bar{z}}}}{f_z}$ is analytic in Δ and satisfies $|a(z)| < 1$ [2]. The function a is called the second complex dilatation function of f. Conversely, any univalent solution of (1.2) with a analytic and $|a| < 1$ is an orientation-preserving harmonic mapping on Δ and can be expressed in the form $f = H \circ D$, where D is a diffeomorphism of Δ and H is analytic in $D(\Delta)$. For more details, see [7].

There is a geometric interpretation of the analytic function a. Let Ω be a doubly connected domain in the extended w-plane having the

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Received October 22, 1994.

1991 AMS Subject Classification: 30C45, 53A10, 30C50.

Key words: harmonic mapping, minimal surface, curvature.

This work was supported by NON DIRECTED RESEARCH FUND, Korea Research Foundation, 1993.

point $w = \infty$ as one of its boundary continua. Consider the nonparametric surface S over Ω whose coordinates are u, v and $G = \phi(u, v)$. Then S is a minimal surface if and only if there is a univalent harmonic mapping $f = h + \overline{g} + A \log |z|$ $(\in \Sigma)$ from Δ onto Ω such that the third component satisfies the differential relation

$$G_z^2 = -\overline{f_{\overline{z}}} f_z = -a f_z^2.$$

The normal direction to the minimal surface at a point (u, v, G) is given by the relation

$$\mathbb{N} = \frac{(-2\mathrm{Im}\{\sqrt{a}\}, -2\mathrm{Re}\{\sqrt{a}\}, 1 - |a|)}{1 + |a|}$$

which depends only on the dilatation function a.

In the second section of this article we discuss the relation between harmonic mappings $f = h + \overline{g} + A \log |z|$ in Δ and the analytic functions $h + \epsilon g$, where $|\epsilon| = 1$, when $\mathbb{C} \backslash f(\Delta)$ is convex or convex in one direction. In the third section we give some applications of univalent harmonic mappings $f \in \Sigma$ to nonparametric minimal surfaces over Ω.

2. Convex and convex in one direction mappings

DEFINITION 2.1. A set D is called convex in the direction ϕ $(0 \leq \phi < \pi)$ if every line parallel to the line through 0 and $e^{i\phi}$ has a connected intersection with D.

DEFINITION 2.2. A function f defined on $\Delta = \{z : |z| > 1\}$ is convex, convex in the direction ϕ $(0 \leq \phi < \pi)$ if f maps Δ onto a domain whose complement is convex, convex in the direction ϕ $(0 \leq \phi < \pi)$, respectively.

DEFINITION 2.3. A function $f(z)$ in a domain D is said to be p-valent in D if for each w_0 (infinity included) the equation $f(z) = w_0$ has at most p roots in D and if there is some w_1 such that the equation $f(z) = w_1$ has exactly p roots in D.

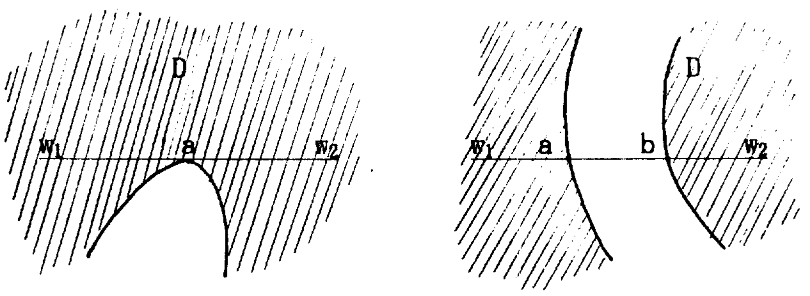
LEMMA 2.4. *Let D $(\infty \in D)$ be a domain whose complement is convex in the direction of the real axis, and let $p(w)$ be a continuous extended real-valued function in D. If the mapping $w \to w + p(w)$ is locally 1-1, then it is at most 2-valent in D. In this case the complement of the image of D is convex in the direction of the real axis.*

Proof. Let $w_1, w_2 \in D$. If $w_1 + p(w_1) = w_2 + p(w_2)$ $(w_1 \neq w_2)$, then writing $w_1 = u_1 + iv_1, w_2 = u_2 + iv_2$ we have $v_1 = v_2 = c$, say, and $u_1 + p(u_1 + ic) = u_2 + p(u_2 + ic)$.

The complement of the image of D is convex in the direction of the real axis, since the mapping $w \to w + p(w)$ maps horizontal lines into themselves.

CASE 1: If $\overline{w_1 w_2} \cap D^c = \phi$, then the real function $u \to u + p(u + ic)$, which is defined on some interval, is not strictly monotonic and therefore not locally 1-1.

CASE 2: If $\overline{w_1 w_2} \cap D^c \neq \phi$, then we have two different cases such that $\overline{w_1 w_2} \cap D^c$ has only one point, say a, or $\overline{w_1 w_2} \cap D^c$ has more than one point (i.e. line segment), say $\overline{ab}$, because D^c is convex in the direction of the real axis (see figure below).



Suppose there exists $w \in \overline{w_1 w_2} \cap D$ such that $w + p(w) = w_1 + p(w_1) = w_2 + p(w_2)$, then $\text{Im}\{w\} = c$. Without loss of generality let's assume $w \in \overline{w_1 a}$, then a contradiction is obtained as in Case 1.

So for each w_0 the equation $w + p(w) = w_0$ has at most 2 roots in D. Therefore $w + p(w)$ is at most 2-valent in D by the definition 2.3.

THEOREM 2.5. *Let $f \in \sum$ and $A \in \mathbb{R}$. If f is convex in the direction of the real axis, then $h - g$ is conformal, at most 2-valent, and convex in the direction of the real axis in Δ.*

Proof. Since f is univalent, there exists a function $z = z(w)$ such that $f(z(w)) = w$ and $z(f(z)) = z$. Thus we have $h - g = f - A\log|z| - 2\mathrm{Re}\{g\}$ and $h(z(w)) - g(z(w)) = w + p(w)$ where $p(w) = -A\log|z(w)| - 2\mathrm{Re}\{g(z(w))\}$. Since $a(z) = \frac{2zg'(z)+A}{2zh'(z)+A}$ satisfies $|a(z)| < 1$, we have $h'(z) - g'(z) \neq 0$ in Δ. Since $h'(z) - g'(z) \neq 0$ and $z(w)$ is 1-1, the mapping $w + p(w)$ is locally 1-1. Therefore $w + p(w)$ is at most 2-valent and convex in the direction of the real axis by Lemma 2.4.

THEOREM 2.6. *Let $f = h + \overline{g} + A\log|z|$, $A \in \mathbb{R}$, be harmonic and locally univalent in Δ. If $h - g$ is a conformal univalent mapping of Δ and convex in the direction of the real axis, then the function f is at most 2-valent and convex in the direction of the real axis.*

Proof. Writing $w = h(z) - g(z)$, $z = z(w)$, we have $f(z(w)) = w + 2\mathrm{Re}\{g(z(w))\} + A\log|z(w)| = w + p(w)$ is locally 1-1 and so at most 2-valent and convex in the direction of the real axis by Lemma 2.4.

THEOREM 2.7. *If a function $f = h + \overline{g} \in \Sigma$ is convex, then the analytic functions*

$$h(z) - e^{i\phi}g(z) \qquad (0 \leq \phi < 2\pi)$$

are convex in the direction $\phi/2$ and at most 2-valent.

Proof. Since f is convex, $e^{-i\phi/2}f = e^{-i\phi/2}\overline{g} + e^{-i\phi/2}h$ is convex. So $e^{-i\phi/2}h - e^{i\phi/2}g$ is convex in the direction of the real axis and at most 2-valent by Theorem 2.5. Hence $h - e^{i\phi}g$ is convex in the direction $\phi/2$ and at most 2-valent.

3. Curvature estimates for some minimal surfaces

Let Ω be a doubly connected domain in the extended w-plane having the point $w = \infty$ as one of its boundary continua. Let S be a nonparametric surface over Ω given by

$$S = \{(u, v, \phi(u, v)) \ : \ u + iv \in \Omega\}.$$

Let

$$\psi(u,v) = \int \frac{\phi_u dv - \phi_v du}{\sqrt{1+\phi_u^2+\phi_v^2}},$$

$$F = \phi + i\psi, \qquad \text{and} \quad \omega = \frac{\phi_u - i\phi_v}{1+\sqrt{1+\phi_u^2+\phi_v^2}}.$$

Then S is a minimal surface if and only if S admits a conformal reparametrization of the form

$$S = \{(u(z), v(z), G(z)) \ : \ z = x+iy \in \Delta\}$$

where

$$u(z) = \mathrm{Re}\Big\{\frac{1}{2}\int F'\Big(\frac{1}{\omega} - \omega\Big)dz\Big\}$$

$$v(z) = \mathrm{Re}\Big\{-\frac{i}{2}\int F'\Big(\frac{1}{\omega} + \omega\Big)dz\Big\}, \qquad \text{and}$$

$$G(z) = \mathrm{Re}\{F(z)\}.$$

The function

$$f = u + iv = \frac{1}{2}\int \frac{F'}{\omega}dz - \frac{1}{2}\overline{\int \omega F' dz} \tag{3.1}$$

is a univalent harmonic mapping from Δ onto Ω with $f(\infty) = \infty$ and G is a real-valued harmonic function satisfying $G_z^2 = -af_z^2 = \omega^2 f_z^2$ where a is defined in (1.2). In Δ the variables F and ω considered as functions of z are regular analytic, $\frac{dF}{dz}$ and ω are single-valued and $\frac{1}{\omega}\frac{dF}{dz} \neq 0, \infty$. The function $\omega(z)$ is regular at $z = \infty$ and $|\omega(\infty)| < 1$. Furthermore, $\frac{F'(z)}{\omega(z)}$ is regular and different from zero at ∞. Observe also that we may assume f is orientation-preserving and that we may obtain any other set of isothermal parameters by applying a conformal mapping to Δ [1]. Since $f(z) \in \Sigma$, f is of the form (1.1).

Remark. The meromorphic function $-1/\omega$ has geometric significance. It is the stereographic projection of the Gauss map. That is, the trace of the unit normal vector to the surface

$$\mathbb{N} = \frac{(-2\mathrm{Re}\{\omega\}, 2\mathrm{Im}\{\omega\},\ 1 - |\omega|^2)}{1+|\omega|^2} \tag{3.2}$$

has stereographic projection $-1/\omega$.

Here is the strategy. Use knowledge about harmonic mappings of Δ onto Ω to gain information about nonparametric minimal surfaces S that lie over Ω.

The Gaussian curvature K at each point of S is given by

$$K = \frac{-16|\omega'|^2}{|\frac{F'}{\omega}|^2(1+|\omega|^2)^4},$$

see [6]. From (3.1) and (1.1), we have that

$$\frac{2(h' - g' + \frac{A-\overline{A}}{2z})}{1+\omega^2} = \frac{F'}{\omega}.$$

Hence,

$$|K| = \frac{4|\omega'|^2|1+\omega^2|^2}{|h' - g' + \frac{A-\overline{A}}{2z}|^2(1+|\omega|^2)^4}.$$

Furthermore the estimate $\frac{|\omega'|}{1-|\omega|^2} \leq \frac{1}{|z|^2-1}$ from Schwarz's lemma for $|z| > 1$ implies

$$|K| \leq \frac{4T(z)}{|h' - g' + \frac{A-\overline{A}}{2z}|^2(|z|^2-1)^2} \tag{3.3}$$

where

$$T(z) = \frac{(1-|\omega|^2)^2|1+\omega^2|^2}{(1+|\omega|^2)^4} \leq 1.$$

In this section we discuss the estimates of $|K(p)|$ for the following two cases:

(1) $\Omega = \mathbb{C}\backslash[a,b]$, arbitrary $p \in \Omega$;
(2) $\Omega = \mathbb{C}\backslash\{0\}$, arbitrary $p \in \Omega$;

where $[a,b]$ is a real line segment in the complex plane.

(1) The case of $\Omega = \mathbb{C}\backslash[a,b]$.

If $\Omega = \mathbb{C}\backslash[a,b]$, then f is of the form

$$f(z) = \alpha z + a_0 + \frac{b_1 + \alpha - \beta}{z} + \overline{\beta z} + \frac{\overline{b_1}}{\overline{z}} + 2\mathrm{Re}\sum_{k=2}^{\infty} \frac{a_k}{z^k} + A\log|z|,$$

where a_0 is real and $\alpha - \beta$ is positive real [4]. So $h - g = (\alpha - \beta)(z + z^{-1}) + a_0$. In [4], the author has observed the following result.

LEMMA 3.1 [4, THEOREM 2.4]. *If S is a nonparametric minimal surface over Ω, and if the unit normal to the surface at ∞ is $(0,0,1)$, then we have*

$$|K(p)| \le \frac{4(b-a)^2}{p_2^2[\pi^2 p_2^2 + 4(b-a)^2]} \quad \text{if } p_2 \neq 0, \tag{3.4}$$

$$|K(p)| \le \frac{4(b-a)^2}{\pi^2 d^4} \quad \text{if } p_2 = 0, \tag{3.5}$$

where $p = p_1 + ip_2 \in \Omega$ and d is the distance from p to $[a,b]$.

REMARK. $\omega(\infty) = 0$ if and only if the unit normal vector to the surface with the standard orientation at ∞ is $(0,0,1)$.

In this article, we want to consider the case $\omega(\infty) \neq 0$. Since ω and F' are analytic in Δ and $\frac{F'}{\omega} \neq 0, \infty$, we have series expansions

$$\omega(z) = \sum_{k=0}^{\infty} \frac{x_k}{z^k}, \qquad F'(z) = \sum_{k=0}^{\infty} \frac{y_k}{z^k}, \tag{3.6}$$

where $x_0 y_0 \neq 0$, $|x_0| < 1$. Substitute these two series into (3.1) and obtain

$$f(z) = \frac{1}{2}\left[\frac{y_0}{x_0} z + \left(\frac{y_1}{x_0} - \frac{y_0 x_1}{x_0^2}\right)\log z + \dots\right] \overline{- \frac{1}{2}\left[x_0 y_0 z + (y_0 x_1 + x_0 y_1)\log z + \dots\right]} + \text{ constant}.$$

Since $f(z)$ must be a single-valued function we must have

$$\frac{y_1}{x_0} - \frac{y_0 x_1}{x_0^2} = -\overline{(y_0 x_1 + x_0 y_1)}.$$

Hence

$$f(z) = \frac{y_0}{2x_0} z + 0(|z|^{-1}) + \overline{\left[-\frac{x_0 y_0}{2} z + 0(|z|^{-1})\right]} + \left(\frac{y_1}{x_0} - \frac{y_0 x_1}{x_0^2}\right) \log|z| + \text{ constant}.$$

That is, we have

$$A = \frac{y_1}{x_0} - \frac{y_0 x_1}{x_0^2} = -\overline{(y_0 x_1 + x_0 y_1)}. \tag{3.7}$$

Also y_1 is real because $\mathrm{Re}\{F\} = \phi$ and ϕ is single-valued.

Let $p = p_1 + ip_2 \in \Omega$; then there exists $z = \gamma e^{i\theta} \in \Delta$ such that $f(z) = p$. Since $\alpha - \beta$ is real, $p_2 = (\alpha - \beta)(\gamma - \gamma^{-1}) \sin\theta + \mathrm{Im}\{A \log \gamma\}$. From (3.3), if A is real we have

$$|K| \leq \frac{4}{\left(\gamma - \frac{1}{\gamma}\right)^4 (\alpha - \beta)^2 + 4p_2^2}. \tag{3.8}$$

The necessary and sufficient condition for $p_2 = 0$ is $\sin\theta = 0$ since $\alpha - \beta$ and $\gamma - \gamma^{-1}$ are positive real for $\gamma > 1$. Thus we obtain the curvature estimates

$$|K| \leq \frac{4}{p_2^2 \left[4 + \frac{p_2^2}{(\alpha-\beta)^2}\right]} \qquad \text{if} \quad p_2 \neq 0$$

and

$$|K| \leq \frac{4}{(\alpha - \beta)^2 \left(\gamma - \frac{1}{\gamma}\right)^4} \qquad \text{if} \quad p_2 = 0.$$

Although the parameter γ in (3.8) is not a geometric quantity, the estimate (3.8) gives us a very important result near $\infty \in \Omega$. Let $\gamma \to \infty$ in the estimate (3.8) to conclude that $K = 0$. After a translation we find that the sum and product of the principal curvatures are zero. This implies that the minimal surface S is plane near ∞ since $f(\infty) = \infty \in \Omega$.

Now we want to discuss what A real means in the case of $\omega(\infty) \neq 0$.

THEOREM 3.2. *A is real if and only if x_0 is real or $x_1 = y_1 = 0$.*

Proof. From (3.7), we obtain

$$y_1 = \frac{x_1 y_0 - \overline{x_1 y_0} x_0^2}{x_0(1 + |x_0|^2)} \tag{3.9}$$

because y_1 is real.

Suppose A is real ; then from (3.7), we have $-(y_0 x_1 + x_0 y_1) = \frac{y_1}{x_0} - \frac{y_0 x_1}{x_0^2}$. Since $0 < |x_0| < 1$, we obtain

$$y_1 = \frac{x_1 y_0(1 - x_0^2)}{x_0(1 + x_0^2)} \quad \text{and} \quad x_1 = \frac{x_0 y_1(1 + x_0^2)}{y_0(1 - x_0^2)}. \tag{3.10}$$

Substitute (3.10) into (3.9) and obtain

$$\frac{2x_0 y_1(1 + |x_0|^2)(|x_0|^2 - x_0^2)}{|1 - x_0^2|^2} = 0.$$

This implies that $y_1 = 0$ or x_0 is real. If $y_1 = 0$, then $x_1 = 0$ from (3.7). Thus A is real implies that $x_1 = y_1 = 0$ or x_0 is real.

Assume that x_0 is real or $x_1 = y_1 = 0$.

If $x_1 = y_1 = 0$, then $A = 0$, so that A is real.

If x_0 is real, then from (3.7) we have $A = -\overline{x_1 y_0} - x_0 y_1$ because y_1 is real. Substitute (3.9) into above equation and obtain $A = \frac{-(\overline{x_1 y_0} + x_1 y_0)}{1 + x_0^2}$. Thus A is real.

The following remarks give some geometric meaning to the conditions in Theorem 3.2.

REMARKS. 1. Let $\mathbb{N}_\infty$ be the unit normal vector to the surface at ∞. Then from (3.2) we have

$$\mathbb{N}_\infty = \left(\frac{-2\mathrm{Re}\{x_0\}}{1 + |x_0|^2}, \frac{2\mathrm{Im}\{x_0\}}{1 + |x_0|^2}, \frac{1 - |x_0|^2}{1 + |x_0|^2} \right).$$

Thus (i) $\mathbb{N}_\infty \subset xz$-plane if and only if x_0 is real.

(ii) The unit normal vector $\mathbb{N}_\infty$ has real stereographic projection $-1/x_0$ if and only if x_0 is real.

2. If $x_1 = y_1 = 0$, then from (3.7) and (3.6), we have $A = 0$ and $F(z) = y_0 z - \frac{y_2}{z} - \frac{y_3}{2z^2} - \ldots +$ constant. Recall that $\mathrm{Re}\{F(z)\} = \phi(u,v)$. We want to find $\phi(u,v)$. Consider the equation

$$w = u + iv = f(z) = \frac{y_0}{2x_0} z - \frac{\overline{x_0 y_0}}{2}\overline{z} + 0(1) \qquad (z \to \infty).$$

Then we obtain

$$\frac{\overline{y}_0}{2\overline{x}_0} w + \frac{\overline{x_0 y_0}}{2}\overline{w} = \frac{|y_0|^2}{4|x_0|^2} z - \frac{|x_0 y_0|^2}{4} z + 0(1).$$

This implies that $z = \frac{2(x_0 w + |x_0|^2 \overline{x_0 w})}{y_0(1-|x_0|^4)} + 0(1)$. Since $\mathrm{Re}\{F(z)\} = \phi(u,v)$, we have

$$\begin{aligned}\phi(u,v) &= \mathrm{Re}\{y_0 z - \frac{y_2}{z} - \frac{y_3}{2z^2} - \ldots + \text{constant}\}\\ &=\mathrm{Re}\left\{ y_0\Big(\frac{2(x_0 w + |x_0|^2\overline{x_0 w})}{y_0(1-|x_0|^4)}\Big) - y_2\Big(\frac{y_0(1-|x_0|^4)}{2(x_0 w + |x_0|^2 \overline{x_0 w})}\Big) - \ldots\right\}.\end{aligned}$$

Thus $\phi(u,v) = \frac{2\mathrm{Re}\{x_0\}}{1-|x_0|^2} u - \frac{2\mathrm{Im}\{x_0\}}{1-|x_0|^2} v + 0(1)$ as $u^2+v^2 \to \infty$. There is no logarithmic term in above equation. Since $y_1 = 0$, $F(z)$ is single-valued function of z, that is, ψ is single valued. By the elementary calculation, we obtain $K = 0$ near $\infty \in \Omega$. So the minimal surface S is plane near ∞.

If $\phi(u,v)$ is interpreted as the potential of a flow of a hypothetical "Chaplygin gas" whose density ρ and speed ν are connected by the relation $\rho^2(1+\nu^2) = 1$, then ψ is the stream-function, F the complex potential and $\phi_u - i\phi_v$ the conjugate complex velocity (cf. [1]).

(2) The case of $\Omega = \{w : |w| > 0\}$.

LEMMA 3.3 [5, THEOREM 2.4]. *If $f \in \Sigma$ and f extends to be of bounded variation on $|z| = 1$, then*

$$|\alpha + \overline{b_1}| \le \frac{L}{2\pi}, \quad |b_n| \le \frac{L}{2n\pi} \quad \textit{for} \quad n \ge 2,$$
$$|\beta + \overline{a_1}| \le \frac{L}{2\pi}, \quad |a_n| \le \frac{L}{2n\pi} \quad \textit{for} \quad n \ge 2,$$

where L is the length of $f(|z|=1)$.

THEOREM 3.4. *If $f \in \Sigma$ and $f(\Delta) = \mathbb{C}\backslash\{p\}$ for some $p \in \mathbb{C}$, then f has the representation*

$$f(z) = \alpha z + p - \frac{\overline{\beta}}{z} + \overline{\beta z} - \frac{\alpha}{\overline{z}} + A\log|z|.$$

Proof. Since $f(\Delta) = \mathbb{C}\backslash\{p\}$, the length of $f(|z| = 1)$ is zero. Therefore it follows from Lemma 3.3 that

$$\alpha = -\overline{b}_1, \quad \beta = -\overline{a}_1, \quad \text{and}$$
$$a_n = b_n = 0 \quad \text{for} \quad n \geq 2.$$

Substitute these into (1.1) and obtain

$$f(z) = \alpha z + a_0 - \frac{\overline{\beta}}{z} + \overline{\beta z} - \frac{\alpha}{\overline{z}} + A\log|z|.$$

$f(|z| = 1) = \{p\}$ implies that $a_0 = p$.

THEOREM 3.5. *If S is a nonparametric minimal surface over $\Omega = \mathbb{C}\backslash\{0\}$ and if the unit normal to the surface at ∞ is $(0, 0, 1)$, then S is a plane.*

Proof. Let $b - a \to 0$ in the curvature estimates (3.4) and (3.5) of Lemma 3.1. Then $K = 0$. Since S is a minimal surface, we find that the sum and product of the principal curvatures are zero at every point. This implies that S is a plane.

References

1. L. Bers, *Isolated singularities of minimal surfaces*, Ann. Math. **53** (1951), 364-386.
2. W. Hengartner and G. Schober, *Harmonic mappings with given dilatation*, J. London Math. Soc. **33** (1986), 473-483.
3. W. Hengartner and G. Schober, *Univalent harmonic functions*, Trans. Amer. Math. Soc. **299** (1987), 1-31.
4. S. H. Jun, *Curvature estimates for minimal surfaces*, Proc. Amer. Math. Soc. **114** (1992), 527-533.
5. S. H. Jun, *Univalent harmonic mappings on $\Delta = \{z : |z| > 1\}$*, Proc. Amer. Math. Soc. **119** (1993), 109-114.
6. R. Osserman, "*A survey of minimal surfaces*", Dover, 1986.

7. W. L. Wendland, *"Elliptic systems in the plane"*, Pitman, London, 1979.

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