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UNIQUENESS OF SQUARE CONVERGENT TRIGONOMETRIC SERIES

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§1. Introduction and the main theorem

It is well known that every periodic function $f \in L^p([0, 2\pi])$, $p > 1$, can be represented by a convergent trigonometric series called the Fourier series of f. Uniqueness of the representing series is very important, and we know that the Fourier series of a periodic function $f \in L^p([0, 2\pi])$ is unique.

More general functions and even distributions can be represented by trigonometric series. The uniqueness problem in this case was first answered by Georg Cantor. He proved that if a trigonometric series converges to 0 everywhere, then all the coeffcients are 0. But, the uniqueness problem for multiple trogonometric series has rather different aspects. For multiple series there are more than one summation ordering. Examples of usually considered summations for 2-dimensional series $\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} a_{m,n}$ are

1) iterated : $\sum_{m=-\infty}^{\infty}(\sum_{n=-\infty}^{\infty} a_{m,n})$,
2) circular : $\lim_{r\to\infty} \sum_{m^2+n^2 \le r^2} a_{m,n}$,
3) rectangular : $\lim_{p,q\to\infty} \sum_{|m|\le p} \sum_{|n|\le q} a_{m,n}$,
4) square : $\lim_{r\to\infty} \sum_{|m|\le r} \sum_{|n|\le r} a_{m,n}$.

If f is a periodic function of n variables in $L^p(\mathbb{T}^n)$, $p > 1$, then the Fourier series of f is square convergent to f a.e.. But, rectangular convergence and circular convergence are not guaranteed. So square convergence seems to be a more natural convergence mode (cf. [F], [S], and [T]).

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For the uniqueness of multiple trigonometric series the following results have been proved :

i) Uniqueness for iterated convergence is an immediate consequence of the Cantor's theorem.

ii) If a double trigonometric series converges circularly to 0 everywhere, then all the coefficients are 0 (cf. [C]).

iii) If a multiple trigonometric series converges rectangularly to 0 everywhere, then all the coefficients are 0 (cf. [AW1], [AW2], and [AFR]).

But, there are no known results about the uniqueness for square convergence. In this paper we give a partial answer to the uniqueness problem for square convergence.

A d-dimensional multiple trigonometric series is a series of the form

$$\sum_{\mathbf{k}\in\mathbb{Z}^d} a_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}},$$

with $\mathbf{x}\in\mathbb{R}^d$. For every nonnegative integer n we let

$$\mathcal{L}_n = \left\{\mathbf{k} = (k_1, \cdots, k_d) \in \mathbb{Z}^d \middle| \max_{1\le j\le d} |k_j| \le n\right\},$$

and

$$\mathcal{B}_n = \begin{cases} \mathcal{L}_0, & \text{if } n = 0 \\ \mathcal{L}_n - \mathcal{L}_{n-1}, & \text{if } n \ge 1. \end{cases}$$

Then

1) $\mathcal{L}_0 \subset \mathcal{L}_1 \subset \mathcal{L}_2 \subset \cdots,$

2) $\mathcal{L}_n = \cup_{k=0}^n \mathcal{B}_k$, for every $n \ge 0$.

A subset E of $\mathbb{Z}^d$ is called a (d-dimensional) *squarelike* indexing set if $\mathcal{L}_{n-1} \subset E \subset \mathcal{L}_n$ for some positive integer n. If $\mathcal{L}_{n-1} \subset E \subset \mathcal{L}_n$ and $E \ne \mathcal{L}_{n-1}$, then the number n is called the order of E and denoted by $|E|$. We also consider $\mathcal{L}_0$ as a squarelike indexing set of order 0. Given multiple series $S = \sum_{\mathbf{k}\in\mathbb{Z}^d} a_{\mathbf{k}}$, a *squarelike* partial sum of S is defined to be $S_E = \sum_{\mathbf{k}\in E} a_{\mathbf{k}}$ for some squarelike indexing set E. We say that $\sum_{\mathbf{k}\in\mathbb{Z}^d} a_{\mathbf{k}}$ converges *squarelikely* to α provided that for every $\epsilon > 0$ there exists N such that if E is a squarelike indexing set and $|E| \ge N$, then $|S_E - \alpha| < \epsilon$. A partial answer to the uniqueness problem for square convergence is now given by the following theorem, which is the main theorem in this paper.

THEOREM. *Suppose that a 2-dimensional trigonometric series*

$$S(x,y) = \sum_{m,n\in\mathbb{Z}} a_{m,n} e^{imx} e^{iny}$$

converges squarelikely to 0 everywhere and its squarelike partial sums $S_E(x,y)$ satisfy $S_E(x,y) = o(|E|^{-1})$ as $|E| \to \infty$. Then all the coefficients $a_{m,n}$ are 0.

In fact, we can prove a similar result for higher dimensional trigonometric series. For that we need the condition that the squarelike partial sums $S_E(\mathbf{x})$ are of $o(|E|^{-d+1})$ as $|E| \to \infty$, where d is the dimension. The proof for this case is quite complicated; while the result is not more satisfactory. So we consider just the 2-dimensional case.

§2. Second formal integral and Schwarz connectors

The second formal integral of a double trigonometric series

$$\sum_{m,n\in\mathbb{Z}} a_{m,n} e^{imx} e^{iny}$$

is a 2-dimensional series

$$\sum_{m,n\in\mathbb{Z}} a^*_{m,n}(x,y) e^{imx} e^{iny},$$

where

$$(1) \qquad a^*_{m,n}(x,y) = \begin{cases} a_{m,n}\dfrac{1}{(im)^2}\dfrac{1}{(in)^2}, & \text{if } m \neq 0 \text{ and } n \neq 0, \\ a_{m,n}(\frac{1}{2}x^2)\dfrac{1}{(in)^2}, & \text{if } m = 0 \text{ and } n \neq 0, \\ a_{m,n}\dfrac{1}{(im)^2}(\dfrac{1}{2}y^2), & \text{if } m \neq 0 \text{ and } n = 0, \\ a_{m,n}(\frac{1}{2}x^2)(\frac{1}{2}y^2), & \text{if } m = n = 0. \end{cases}$$

PROPOSITION 1. *Let* $S(x,y)=\sum_{m,n\in\mathbb{Z}} a_{m,n}e^{imx}e^{iny}$. *If* $S=\sum_{m,n\in\mathbb{Z}} a_{m,n}$ *converges squarelikely, then the second formal integral of* $S(x,y)$ *converges absolutely and uniformly to a continuous function.*

Proof. By the hypothesis, the squarelike partial sums of S are bounded. Choose $M>0$ such that $|S_E|\leq M/2$ for every squarelike indexing set E in $\mathbb{Z}^2$. Since

$$a_{m,n}=\begin{cases} S_{\mathcal{L}_{|m|-1}\cup\{(m,n)\}}-S_{\mathcal{L}_{|m|-1}}, & \text{if } |m|\geq |n|,\\ S_{\mathcal{L}_{|n|-1}\cup\{(m,n)\}}-S_{\mathcal{L}_{|n|-1}}, & \text{if } |m|<|n|,\end{cases}$$

we have $|a_{m,n}|\leq M$ for all $m,n\in\mathbb{Z}$. Hence

$$\begin{aligned}\sum_{m,n\in\mathbb{Z}}\left|a^*_{m,n}(x,y)e^{imx}e^{iny}\right| &= \sum_{m,n\in\mathbb{Z}}\left|a^*_{m,n}(x,y)\right|\\ &\leq M\left[\sum_{m\neq 0}\sum_{n\neq 0}\frac{1}{m^2n^2}+\frac{x^2}{2}\sum_{n\neq 0}\frac{1}{n^2}+\frac{y^2}{2}\sum_{m\neq 0}\frac{1}{m^2}+\frac{x^2y^2}{4}\right]\\ &<\infty.\end{aligned}$$

The assertion now follows. □

We now define a well-ordering $\prec$ on $\mathbb{Z}^2$ for which the immediate successor and predecessor of (m,n) are denoted by $(m,n)^+$ and $(m,n)^-$, respectively. Consider $(0,0)$ as the smallest element, and for each $(m,n)\in\mathbb{Z}^2$ define

$$(2)\qquad (m,n)^+=\begin{cases}(0,-1), & \text{if } m=n=0,\\ (m+1,n), & \text{if } n<0 \text{ and } n<m<-n,\\ (m,n+1), & \text{if } m>0 \text{ and } -m\leq n<m,\\ (m-1,n), & \text{if } n>0 \text{ and } -n<m\leq n,\\ (m,n-1), & \text{if } m<0 \text{ and } m\leq n\leq -m.\end{cases}$$

We thus have an ordering on $\mathbb{Z}^2$ so that

$$(3)\qquad \begin{aligned}&(0,0)\prec(0,-1)\prec(1,-1)\prec(1,0)\prec(1,1)\prec(0,1)\prec(-1,1)\prec(-1,0)\\ &\quad\prec(-1,-1)\prec(-1,-2)\prec(0,-2)\prec(1,-2)\prec(2,-2)\prec(2,-1)\prec\cdots.\end{aligned}$$

One can imagine this sequence as an outward spiral which traces $\mathbb{Z}^2$ counterclockwise with starting point $(0,0)$. Note that this sequence fills up $\mathcal{B}_n$ successively. Hence, we have

$$
\text{(4)}\quad \begin{aligned}
&\text{i)}\quad \mathcal{L}_N = \{(m,n)\in\mathbb{Z}^2 \,|\, (m,n)\preceq(-N,-N)\} \quad \text{for} \quad N\geq 0,\\
&\text{ii)}\quad \mathcal{B}_N = \{(m,n)\in\mathbb{Z}^2 \,|\, (-N+1,-N)\preceq(m,n)\preceq(-N,-N)\}\\
&\qquad \text{for } N>0,
\end{aligned}
$$

and for each $(p,q)\in\mathbb{Z}^2$ the sum over the indexing set $\{(m,n)|(m,n)\preceq(p,q)\}$ defines a *squarelike* partial sum.

PROPOSITION 2. *Let* $A_{(p,q)} = \sum_{(m,n)\preceq(p,q)} a_{(m,n)}$ *and* $\Delta b_{(p,q)} = b_{(p,q)} - b_{(p,q)^+}$ *for* $(p,q)\in\mathbb{Z}^2$. *Then for* $N>0$

$$
\text{i)}\quad \sum_{(m,n)\in\mathcal{B}_N} a_{(m,n)}b_{(m,n)} = \sum_{(m,n)\in\mathcal{B}_N} A_{(m,n)}\Delta b_{(m,n)} + A_{(-N,-N)}b_{(-N,-N)^+} - A_{(-N+1,-N+1)}b_{(-N,-N)^+},
$$

and for $N\geq 0$

$$
\text{ii)}\quad \sum_{(m,n)\in\mathcal{L}_N} a_{(m,n)}b_{(m,n)} = \sum_{0\leq k\leq N}\sum_{(m,n)\in\mathcal{B}_k} A_{(m,n)}\Delta b_{(m,n)} + A_{(-N,-N)}b_{(-N,-N)^+}.
$$

Proof. From the definition of $A_{(m,n)}$ we have

$$
\begin{aligned}
\sum_{(m,n)\in\mathcal{B}_N} a_{(m,n)}b_{(m,n)} &= \sum_{(m,n)\in\mathcal{B}_N} \left(A_{(m,n)} - A_{(m,n)^-}\right)b_{(m,n)}\\
&= \sum_{(m,n)\in\mathcal{B}_N} A_{(m,n)}b_{(m,n)} - \sum_{(m,n)\in\mathcal{B}_N} A_{(m,n)^-}b_{(m,n)}.
\end{aligned}
$$

From $\mathcal{B}_N = \{(m,n)\in\mathbb{Z}^2 \,|\, (-N+1,-N)\preceq(m,n)\preceq(-N,-N)\}$ for

$N > 0$, we have

$$\begin{aligned}\sum_{(m,n)\in\mathcal{B}_N} A_{(m,n)^-}b_{(m,n)} &= \sum_{(-N+1,-N)\preceq(m,n)\preceq(-N,-N)} A_{(m,n)^-}b_{(m,n)} \\ &= \sum_{(-N+1,-N+1)\preceq(m,n)\preceq(-N,-N+1)} A_{(m,n)}b_{(m,n)^+} \\ &= \sum_{(m,n)\in\mathcal{B}_N} A_{(m,n)}b_{(m,n)^+} + A_{(-N+1,-N+1)}b_{(-N+1,-N+1)^+} \\ &\qquad - A_{(-N,-N)}b_{(-N,-N)^+}.\end{aligned}$$

Substituting the last expression into the above equation, we obtain i). Since $a_{(0,0)} = A_{(0,0)}$, i) then implies

$$\begin{aligned}&\sum_{(m,n)\in\mathcal{L}_N} a_{(m,n)}b_{(m,n)} = \sum_{1\le k\le N}\sum_{(m,n)\in\mathcal{B}_k} a_{(m,n)}b_{(m,n)} + A_{(0,0)}b_{(0,0)} \\ &= \sum_{1\le k\le N}\Big(\sum_{(m,n)\in\mathcal{B}_k} A_{(m,n)}\Delta b_{(m,n)} + A_{(-k,-k)}b_{(-k,-k)^+} \\ &\qquad - A_{(-k+1,-k+1)}b_{(-k+1,-k+1)^+}\Big) + A_{(0,0)}b_{(0,0)} \\ &= \sum_{1\le k\le N}\sum_{(m,n)\in\mathcal{B}_k} A_{(m,n)}\Delta b_{(m,n)} \\ &+ \left[\sum_{1\le k\le N}\left(A_{(-k,-k)}b_{(-k,-k)^+} - A_{(-k+1,-k+1)}b_{(-k+1,-k+1)^+}\right) + A_{(0,0)}b_{(0,0)}\right].\end{aligned}$$

The expression in the bracket is equal to $A_{(-N,-N)}b_{(-N,-N)^+}$. We thus obtain ii) and completes the proof. □

For $j = 1,2$ let $\mathbf{e}_j$ be the unit vector in $\mathbb{R}^2$ whose j-th coordinate is 1, and for $j = 1,2$, $h \in \mathbb{R}$, and $\mathbf{x} \in \mathbb{R}^2$, let

$$\Delta^0_{j,h} f(\mathbf{x}) = f(\mathbf{x}) - 2f(\mathbf{x} + h\mathbf{e}_j) + f(\mathbf{x} + 2h\mathbf{e}_j),$$

and

$$\Delta^1_{j,h} f(\mathbf{x}) = f(\mathbf{x} - h\mathbf{e}_j) - 2f(\mathbf{x}) + f(\mathbf{x} + h\mathbf{e}_j).$$

The Schwarz difference $\Delta^{s,t}_{(h,k)} f(\mathbf{x})$ and difference quotient $Q^{s,t}_{(h,k)} f(\mathbf{x})$ is defined by

$$\Delta^{s,t}_{(h,k)} f(\mathbf{x}) = \Delta^s_{1,h}\Delta^t_{2,k} f(\mathbf{x}) \quad \text{and} \quad Q^{s,t}_{(h,k)} f(\mathbf{x}) = \frac{\Delta^{s,t}_{(h,k)} f(\mathbf{x})}{(h^2)^s(k^2)^t}.$$

for $(s,t) \in \{0,1\}^2$, $(h,k) \in \mathbb{R}^2 - \{(0,0)\}$, and $\mathbf{x} \in \mathbb{R}^2$. We now define the Schwarz connector $D^{s,t} f(\mathbf{x})$ by

$$D^{s,t} f(\mathbf{x}) = \lim_{h,k \to 0} Q^{s,t}_{(h,k)} f(\mathbf{x}),$$

where the limit is taken over such $(h,k) \in \mathbb{R}^2 - \{(0,0)\}$ as $1/2 \leq |h/k| \leq 2$.

PROPOSITION 3. *Suppose that the second formal integral of the double trigonometric series $\sum_{m,n\in\mathbb{Z}} a_{m,n} e^{imx} e^{iny}$ converges everywhere to a function $f(x,y)$. Then for each $(s,t) \in \{0,1\}^2$ the Schwarz difference quotient of $f(x,y)$ is given by*

$$Q^{s,t}_{(h,k)} f(x,y) = \sum_{m,n\in\mathbb{Z}} a_{m,n} e^{imx} e^{iny} \lambda_{s,m}(h) \lambda_{t,n}(k), \tag{5}$$

where

$$\lambda_{1,k}(r) = \begin{cases} \left(\dfrac{\sin(kr/2)}{kr/2}\right)^2, & \text{if } k \neq 0, \\ 1, & \text{if } k = 0, \end{cases}$$

and $\lambda_{0,k}(r) = \lambda_{1,k}(r) r^2 e^{ikr}$.

Proof. Since $f(x,y) = \sum_{m,n\in\mathbb{Z}} a^*_{m,n}(x,y) e^{imx} e^{iny}$, where $a^*_{m,n}(x,y)$ are given by (1), we have

$$Q^{s,t}_{(h,k)} f(x,y) = \sum_{m,n\in\mathbb{Z}} Q^{s,t}_{(h,k)} \big(a^*_{m,n}(x,y) e^{imx} e^{iny}\big).$$

But, by the definition of the Schwarz difference, we have

$$Q^{s,t}_{(h,k)} \big(a^*_{m,n}(x,y) e^{imx} e^{iny}\big)$$

$$= \begin{cases} a_{m,n} \dfrac{\Delta_{s,h}(e^{imx})}{(im)^2 (h^2)^s} \dfrac{\Delta_{t,k}(e^{iny})}{(in)^2 (k^2)^t}, & \text{if } m \neq 0, n \neq 0, \\ a_{m,n} \dfrac{\Delta_{s,h}(\frac{1}{2}x^2)}{(h^2)^s} \dfrac{\Delta_{t,k}(e^{iny})}{(in)^2 (k^2)^t}, & \text{if } m = 0, n \neq 0, \\ a_{m,n} \dfrac{\Delta_{s,h}(e^{imx})}{(im)^2 (h^2)^s} \dfrac{\Delta_{t,k}(\frac{1}{2}y^2)}{(k^2)^t}, & \text{if } m \neq 0, n = 0, \\ a_{m,n} \dfrac{\Delta_{s,h}(\frac{1}{2}x^2)}{(h^2)^s} \dfrac{\Delta_{t,k}(\frac{1}{2}y^2)}{(k^2)^t}, & \text{if } m = n = 0. \end{cases} \tag{6}$$

It is easy to check that

$$(7)\qquad \frac{\Delta_{p,r}(e^{ijw})}{(ij)^2(r^2)^p} = \begin{cases} e^{ijw}\lambda_{1,j}(r), & \text{if } p=1, \\ \\ e^{ijw}\lambda_{0,j}(r), & \text{if } p=0, \end{cases}$$

for $j \neq 0$, and

$$(8)\qquad \frac{\Delta_{p,r}(\frac{1}{2}w^2)}{(r^2)^p} = \begin{cases} \lambda_{1,0}(r), & \text{if } p=1, \\ \\ \lambda_{0,0}(r), & \text{if } p=0. \end{cases}$$

Replacing $\frac{\Delta_{p,r}(e^{ijw})}{(ij)^2(r^2)^p}$ and $\frac{\Delta_{p,r}(\frac{1}{2}w^2)}{(r^2)^p}$ in (6) with (7) and (8), we now have

$$Q^{s,t}_{(h,k)}\big(a^*_{m,n}(x,y)e^{imx}e^{iny}\big) = a_{m,n}e^{imx}e^{iny}\lambda_{s,m}(h)\lambda_{t,n}(k),$$

and completes the proof. □

LEMMA 4. *Let* $\mu_m(h) = \sin^2(mh/2)/\big(mh/2\big)^2$ *and* $\Delta\mu_m(h) = \mu_{m+1}(h) - \mu_m(h)$. *Then, there are positive constants* C_1 *and* C_2 *such that if* $0 < |h| < 1$, *then*

$$(9)\qquad \sum_{|m|\leq N} \big|\Delta\mu_m(h)\big| \leq C_1 N^2 h^2,$$

for every N *with* $1 \leq N \leq 1/|h|$, *and*

$$(10)\qquad \sum_{|m|\leq N} \big|\Delta\mu_m(h)\big| \leq C_2,$$

for every positive integer N.

Proof. Let us define $\theta(t) = \big(\sin(t/2)\big)/\big(t/2\big)$ if $t \neq 0$, and $\theta(0) = 1$. It is easy to check that $|\theta(t)| \leq 1$ and $|\theta(t)| \leq 2/|t|$ for every t. We can also find two constants C_3 and C_4 such that

$$|\theta'(t)| \leq \begin{cases} C_3|t|, & \text{if } |t| \leq 2, \\ \\ C_4\frac{1}{|t|}, & \text{if } |t| \geq \frac{1}{2}. \end{cases}$$

Since

$$\Delta\mu_m(h) = \big(\theta((m+1)h)\big)^2 - \big(\theta(mh)\big)^2$$
$$= \big(\theta((m+1)h) + \theta(mh)\big)\big(\theta((m+1)h) - \theta(mh)\big),$$

we have

$$|\Delta\mu_m(h)| \leq 2\big(\theta((m+1)h) - \theta(mh)\big) = 2\Big|\int_{mh}^{(m+1)h} \theta'(t)dt\Big|.$$

Hence, if $0 < |h| < 1$ and $1 \leq N \leq 1/|h|$, then

$$\sum_{|m|\leq N} |\Delta\mu_m(h)| \leq 2 \sum_{|m|\leq N} \Big|\int_{mh}^{(m+1)h} \theta'(t)dt\Big|$$
$$\leq 2\int_{-(N+1)|h|}^{(N+1)|h|} |\theta'(t)|dt$$
$$\leq 4C_3 \int_0^{2N|h|} tdt = 8C_3N^2h^2,$$

and so we obtain (9) with $C_1 = 8C_3$. In particular, $\sum_{|m|\leq\frac{1}{|h|}} |\Delta\mu_m(h)| \leq C_1$. So in order to obtain (10), it suffices to show $\sum_{|m|>\frac{1}{|h|}} |\Delta\mu_m(h)| \leq C$ for some constant C. But, if $|m| > \frac{1}{|h|}$, then

$$|\theta((m+1)h) + \theta(mh)| \leq \frac{2}{|(m+1)h|} + \frac{2}{|mh|} \leq \frac{6}{|mh|}$$

and

$$|\theta((m+1)h) - \theta(mh)| \leq \sup_{t\in[mh,(m+1)h]} |\theta'(t)| \cdot |h| \leq \frac{2C_4}{|m|}.$$

Hence, if $0 < |h| < 1$, then

$$\sum_{|m|>\frac{1}{|h|}} |\Delta\mu_m(h)| \leq \frac{24C_4}{|h|} \sum_{m>\frac{1}{|h|}} \frac{1}{m^2} \leq \frac{24C_4}{|h|}\left(h^2 + \int_{\frac{1}{|h|}}^{\infty} \frac{1}{t^2}dt\right)$$
$$= 24C_4(|h|+1) <= 48C_4.$$

The proof is now completed. □

LEMMA 5. *For each* $(s,t) \in \{0,1\}^2$ *and* $m, n \in \mathbb{Z}$, *let* $\lambda^{s,t}_{(m,n)}(h,k) = \lambda_{s,m}(h)\lambda_{t,n}(k)$ *and* $\Delta\lambda^{s,t}_{(m,n)}(h,k) = \lambda^{s,t}_{(m,n)}(h,k) - \lambda^{s,t}_{(m,n)+}(h,k)$, *where* $\lambda_{s,m}$ *and* $\lambda_{t,n}$ *are defined as in Proposition 3. Then there is a constant* $C > 0$ *such that*

$$\sum_{N \geq 1} \frac{1}{N} \sum_{(m,n)\in\mathcal{B}_N} \left|\Delta\lambda^{s,t}_{(m,n)}(h,k)\right| \leq C, \tag{11}$$

for all $(h,k) \in \mathbb{R}^2$ *with* $0 < |h| < 1, 0 < |k| < 1$, *and* $1/2 \leq |h/k| \leq 2$.

Proof. Consider $\sum_{(m,n)\in\mathcal{B}_N} \left|\Delta\lambda^{s,t}_{(m,n)}(h,k)\right|$. By the definition of the ordering $\prec$, we can write

$$\begin{aligned}
&\sum_{(m,n)\in\mathcal{B}_N} \left|\Delta\lambda^{s,t}_{(m,n)}(h,k)\right| \\
&= \sum_{m=-N+1}^{N-1} \left|\lambda^{s,t}_{(m,-N)}(h,k) - \lambda^{s,t}_{(m+1,-N)}(h,k)\right| \\
&\quad + \sum_{n=-N}^{N-1} \left|\lambda^{s,t}_{(N,n)}(h,k) - \lambda^{s,t}_{(N,n+1)}(h,k)\right| \\
&\quad + \sum_{m=-N+1}^{N} \left|\lambda^{s,t}_{(m,N)}(h,k) - \lambda^{s,t}_{(m-1,N)}(h,k)\right| \\
&\quad + \sum_{n=-N}^{N} \left|\lambda^{s,t}_{(-N,n)}(h,k) - \lambda^{s,t}_{(-N,n-1)}(h,k)\right| \\
&= \Big(\sum_{m=-N+1}^{N-1} \left|\lambda_{s,m+1}(h) - \lambda_{s,m}(h)\right|\Big) \left|\lambda_{t,-N}(k)\right| \\
&\quad + \Big(\sum_{m=-N}^{N-1} \left|\lambda_{t,n+1}(k) - \lambda_{t,n}(k)\right|\Big) \left|\lambda_{s,N}(h)\right| \\
&\quad + \Big(\sum_{m=-N+1}^{N} \left|\lambda_{s,m}(h) - \lambda_{s,m-1}(h)\right|\Big) \left|\lambda_{t,N}(k)\right| \\
&\quad + \Big(\sum_{m=-N}^{N} \left|\lambda_{t,n}(k) - \lambda_{t,n-1}(k)\right|\Big) \left|\lambda_{s,-N}(h)\right|.
\end{aligned}$$

Since $|\lambda_{u,-N}(r)| = |\lambda_{u,N}(r)|$ for $u = 0, 1$, we have

$$\sum_{(m,n)\in\mathcal{B}_N} \left|\Delta\lambda^{s,t}_{(m,n)}(h,k)\right| \le 2\Big(\sum_{|m|\le 2N} \left|\lambda_{s,m+1}(h) - \lambda_{s,m}(h)\right|\Big)\left|\lambda_{t,N}(k)\right| + 2\Big(\sum_{|n|\le 2N} \left|\lambda_{t,n+1}(k) - \lambda_{t,n}(k)\right|\Big)\left|\lambda_{s,N}(h)\right|.$$

Let $F^{s,t}(h,k) = \sum_{N\ge 1}\sum_{|m|\le 2N} |\lambda_{s,m+1}(h) - \lambda_{s,m}(h)||\lambda_{t,N}(k)|$. Then,

$$\sum_{N\ge 1}\frac{1}{N}\sum_{(m,n)\in\mathcal{B}_N} \left|\Delta\lambda^{s,t}_{(m,n)}(h,k)\right| \le 2\big(F^{s,t}(h,k) + F^{t,s}(k,h)\big). \tag{12}$$

Now suppose $0 < |h| < 1, 0 < |k| < 1$, and $1/2 \le |h/k| \le 2$. Then,

$$|\lambda_{0,N}(k)| \le |\lambda_{1,N}(k)| = \mu_N(k),$$

and

$$|\lambda_{1,m+1}(h) - \lambda_{1,m}(h)| = |\Delta\mu_m(h)|,$$

where $\mu_N(k)$ and $\Delta\mu_m(h)$ are defined as in Lemma 4. So we have

$$F^{1,t}(h,k) \le \sum_{N\ge 1}\frac{1}{N}\sum_{|m|\le 2N} |\Delta\mu_m(h)||\mu_N(k)|.$$

Hence, from $|\mu_N(k)| \le 1$ and (9) it follows that

$$\sum_{1\le N\le\frac{1}{2|h|}}\frac{1}{N}\sum_{|m|\le 2N} |\Delta\mu_m(h)||\mu_N(k)| \le \sum_{1\le N\le\frac{1}{2|h|}}\frac{1}{N}\cdot C_1(2N)^2h^2 \le C_1.$$

On the other hand, from $|\mu_N(k)| \le \frac{4}{N^2k^2} \le \frac{16}{N^2h^2}$ and (10) it follows that

$$\sum_{N>\frac{1}{2|h|}}\frac{1}{N}\sum_{|m|\le 2N} |\Delta\mu_m(h)||\mu_N(k)| \le \sum_{N>\frac{1}{2|h|}}\frac{1}{N}\cdot C_2\frac{16}{N^2h^2} \le 232C_2.$$

We thus have $F^{1,t}(h,k) \le C$ for some constant C. By the same argument we also have $F^{1,s}(k,h) \le C$.

Now consider $F^{0,t}(h,k)$ and $F^{0,s}(k,h)$. Since $|\lambda_{0,m}(h)| \leq h^2$ and $\lambda_{0,0}(h) = 0$,

$$\sum_{|m|\leq 2N} \left|\lambda_{s,m+1}(h) - \lambda_{s,m}(h)\right| \leq 8Nh^2,$$

and so

$$\begin{aligned} F^{0,t}(h,k) \leq & \sum_{N\geq 1} \frac{1}{N} \cdot 8Nh^2 \cdot |\mu_N(k)| \leq \sum_{N\geq} \frac{1}{N} \cdot 8Nh^2 \cdot \frac{4}{N^2k^2} \\ =& 128 \sum_{N\geq 1} \frac{1}{N^2} < C \end{aligned}$$

for some constant C, and by a similar argument $F^{0,s}(k,h) \leq C$. Putting these result together into (12), we can now derive (11). □

The following proposition is the main result in this section.

PROPOSITION 6. *Suppose that a 2-dimensional trigonomeric series*

$$S(x,y) = \sum_{m,n\in\mathbb{Z}} a_{m,n} e^{imx} e^{iny}$$

converges squarelikely to 0 everywhere, and that all of its squarelike partial sums $S_E(x,y)$ *satisfy* $S_E(x,y) = o(|E|^{-1})$ *as* $|E| \to \infty$. *Let* $f(x,y)$ *be the second formal integral of* $S(x,y)$. *Then, for each* $(s,t) \in \{0,1\}^2$ *the Schwarz connector* $D^{s,t}f(x,y)$ *is identically 0.*

Proof. Since $S(0,0) = \sum_{m,n\in\mathbb{Z}} a_{m,n}$ converges squarelikely, from proposition 1 it follows that the series for $f(x,y)$ converges absolutely. Hence, for nonzero reals h and k the series for the Schwarz difference quotient $Q^{s,t}_{(h,k)}f(x,y)$ converges absolutely. But, as in (5), $Q^{s,t}_{(h,k)}f(x,y)$ is given by

$$Q^{s,t}_{(h,k)}f(x,y) = \sum_{m,n\in\mathbb{Z}} a_{m,n} e^{imx} e^{iny} \lambda_{s,m}(h)\lambda_{t,n}(k).$$

By the absolute convergence, we can rewrite

$$Q^{s,t}_{(h,k)}f(x,y) = \lim_{N\to\infty} P^{s,t}_N(x,y),$$

where $P_N^{s,t}(x,y) = \sum_{(m,n)\in\mathcal{L}_N} a_{m,n}e^{imx}e^{iny}\lambda_{s,m}(h)\lambda_{t,n}(k)$. Put

$$\lambda_{(m,n)}^{s,t}(h,k) = \lambda_{s,m}(h)\lambda_{t,n}(k).$$

Then, by the summation by parts formula proved in Proposition 2, we have

$$P_N^{s,t}(x,y) = \sum_{0\le p\le N}\sum_{(m,n)\in\mathcal{B}_p} A_{(m,n)}(x,y)\Delta\lambda_{(m,n)}^{s,t}(h,k) + A_{(-N,-N)}(x,y)\lambda_{(-N,-N)^+}^{s,t}(h,k),$$

where

$$A_{(m,n)}(x,y) = \sum_{(p,q)\preceq(m,n)} a_{m,n}e^{imx}e^{iny},$$

and

$$\Delta\lambda_{(m,n)}^{s,t}(h,k) = \lambda_{(m,n)}^{s,t}(h,k) - \lambda_{(m,n)^+}^{s,t}(h,k).$$

Note that each $A_{(m,n)}(x,y)$ is a squarelike partial sum of $S(x,y)$, and so

$$\lim_{N\to\infty} A_{(-N,-N)}(x,y) = 0$$

by the hypothesis. Since $\lambda_{(m,n)}^{s,t}(h,k)$ are bounded for small h and k, we thus have

$$Q_{(h,k)}^{s,t}f(x,y) = \sum_{N\ge 0}\sum_{(m,n)\in\mathcal{B}_N} A_{(m,n)}(x,y)\Delta\lambda_{(m,n)}^{s,t}(h,k). \tag{13}$$

Now fix $(s,t)\in\{0,1\}^2$ and $(x,y)\in\mathbb{R}^2$, and let $\epsilon>0$ be given arbitrarily. By the hypothesis about the squarelike partial sums, we can choose $M>0$ such that

$$|A_{(m,n)}(x,y)| \le \frac{\epsilon}{N} \tag{14}$$

whenever $(m,n)\in\mathcal{B}_N$ and $N\ge M$. Since $\lim_{h,k\to 0}\Delta\lambda_{(m,n)}^{s,t}(h,k) = 0$,

$$\lim_{h,k\to 0}\sum_{0\le N\le M}\sum_{(m,n)\in\mathcal{B}_N} A_{(m,n)}(x,y)\Delta\lambda_{(m,n)}^{s,t}(h,k) = 0.$$

Hence we can choose a positive number η with $\eta < 1/M$ such that

$$(15) \qquad \Big| \sum_{0 \le N \le M} \sum_{(m,n) \in \mathcal{B}_N} A_{(m,n)}(x,y) \Delta\lambda^{s,t}_{(m,n)}(h,k) \Big| < \epsilon$$

for all h and k with $|h| \le \eta$ and $|k| \le \eta$. On the other hand, by (14) we have

$$\begin{aligned} \Big| \sum_{N > M} \sum_{(m,n) \in \mathcal{B}_N} & A_{(m,n)}(x,y) \Delta\lambda^{s,t}_{(m,n)}(h,k) \Big| \\ &\le \sum_{N > M} \sum_{(m,n) \in \mathcal{B}_N} \big| A_{(m,n)}(x,y) \big| \big| \Delta\lambda^{s,t}_{(m,n)}(h,k) \big| \\ &\le \epsilon \sum_{N > M} \frac{1}{N} \sum_{(m,n) \in \mathcal{B}_N} \big| \Delta\lambda^{s,t}_{(m,n)}(h,k) \big| \\ &\le \epsilon \sum_{N \ge 1} \frac{1}{N} \sum_{(m,n) \in \mathcal{B}_N} \big| \Delta\lambda^{s,t}_{(m,n)}(h,k) \big| \le C\epsilon. \end{aligned}$$

The last inequality follows from lemma 5. This result together with (15) implies

$$\big| Q^{s,t}_{(h,k)} f(x,y) \big| \le C\epsilon$$

provided $0 < |h| < \eta$, $0 < |k| < \eta$, and $1/2 \le |h/k| \le 2$. Since ϵ was arbitrary, we therefore conclude $D^{s,t} f(x,y) = \lim_{h,k \to 0} Q^{s,t}_{(h,k)} f(x,y) = 0$. □

§3. Proof of the main theorem

We first quote the following result from [AFR]. For the original version of the following proposition and the detailed proof the readers are referred to [AFR].

PROPOSITION 7 [AFR]. *Let $f(x,y)$ be a continuous function. If all the Schwarz connectors of $f(x,y)$ are identically 0, then all the Schwarz differences of $f(x,y)$ are identically 0.*

Suppose that $S(x,y) = \sum_{m,n \in \mathbb{Z}} a_{m,n} e^{imx} e^{iny}$ converges squarelikely to 0 everywhere and the squarelike partial sums $A_{(p,q)} = \sum_{(m,n) \preceq (p,q)}$

$a_{m,n}e^{imx}e^{iny}$, $(p,q) \in \mathcal{B}_N$, are equal to $o(N^{-1})$ as $N \to \infty$. Then, proposition 1 implies that the second formal integral $f(x,y)$ of $S(x,y)$ is continuous, and proposition 6 implies that all Schwarz connectors of $f(x,y)$ are identically 0. From proposition 7 it now follows that $\Delta^{s,t}_{(h,k)} f(x,y) = 0$ for all $(s,t) \in \{0,1\}^2$ and $(h,k) \in \mathbb{R}^2 - \{(0,0)\}$. In particular, $\Delta^{1,1}_{(2\pi,2\pi)} f(x,y) = 0$. But, $\Delta^{1,1}_{(2\pi,2\pi)} f(x,y) = a_{0,0}$. Hence we obtain the following result.

COROLLARY 8. *Suppose that a 2-dimensional trigonometric series*

$$S(x,y) = \sum_{m,n\in\mathbb{Z}} a_{m,n}e^{imx}e^{iny}$$

converges squarelikely to 0 everywhere and all of its squarelike partial sums $S_E(x,y)$ are equal to $o(|E|^{-1})$ as $|E| \to \infty$. Then $a_{0,0} = 0$.

For each $(p,q) \in \mathbb{Z}^2$ let us define a transformation $\tau_{(p,q)}$ which translates a 2-dimensional trigonometric series :

$$\tau_{(p,q)}\Bigl(\sum_{m,n\in\mathbb{Z}} a_{m,n}e^{imx}e^{iny}\Bigr) = \sum_{m,n\in\mathbb{Z}} a_{m-p,n-q}e^{imx}e^{iny}.$$

PROPOSITION 9. *If $S(x,y) = \sum_{m,n\in\mathbb{Z}} a_{m,n}e^{imx}e^{iny}$ satisfies the hypothesis of the main theorem, then $\tau_{(p,q)}S(x,y)$ also satisfies the same hypothesis for every $(p,q) \in \mathbb{Z}^2$.*

Proof. Once we prove the assertion for the cases of $(p,q) = (\pm 1, 0)$, $(0,\pm 1)$, then by the repeat of the same argument we can obtain the desired result. The treatments of the cases of $(p,q) = (\pm 1,0), (0,\pm 1)$ are similar to each other. So we shall consider only the case of $(p,q) = (1,0)$.

Put

$$T(x,y) = \tau_{(1,0)}S(x,y) = \sum_{m,n\in\mathbb{Z}} b_{m,n}e^{imx}e^{iny},$$

where $b_{m,n} = a_{m-1,n}$, and for each $E \subset \mathbb{Z}^2$ write $S_E(x,y) = \sum_{(m,n)\in E} a_{m,n}e^{imx}e^{iny}$ and $T_E(x,y) = \sum_{(m,n)\in E} b_{m,n}e^{imx}e^{iny}$. Let $\epsilon > 0$ be given arbitrarily, and choose $M > 0$ with $M - 1 \geq M/2$ such that

$$(16) \qquad \bigl|S_E(x,y)\bigr| < \frac{\epsilon}{N} \quad \text{if} \quad N \geq M \quad \text{and} \quad \mathcal{L}_{N-1} \subset E \subset \mathcal{L}_N.$$

Suppose $N \geq M$ and $\mathcal{L}_{N-1} \subset E \subset \mathcal{L}_N$, and let $F = \{(m,n) \in \mathbb{Z}^2 | (m+1,n) \in E\}$. Then

$$T_E(x,y) = \sum_{(m,n)\in E} b_{m,n} e^{imx} e^{iny} = \sum_{(m,n)\in E} a_{m-1,n} e^{imx} e^{iny}$$
$$= e^{ix} \sum_{(m,n)\in F} a_{m,n} e^{imx} e^{iny} = e^{ix} S_F(x,y).$$

Clearly $\mathcal{L}_{N-2} \subset F \subset \mathcal{L}_{N+1}$.

Now put $A = F \cap \mathcal{L}_{N-1}, B = \mathcal{L}_{N-1} \cup (F \cap \mathcal{B}_N)$, and $C = \mathcal{L}_N \cup (F \cap \mathcal{B}_{N+1})$. Then

(17)
$$T_E(x,y) = e^{ix}\Big(S_A(x,y)+S_B(x,y)+S_C(x,y)-S_{\mathcal{L}_{N-1}}(x,y)-S_{\mathcal{L}_N}(x,y)\Big).$$

But by (16),

$$|S_A(x,y)| < \epsilon/(N-1) \leq 2\epsilon/N,$$
$$|S_B(x,y)| < \epsilon/N, \quad |S_C(x,y)| < \epsilon/(N+1) < \epsilon/N,$$

and

$$|S_{\mathcal{L}_{N-1}}(x,y)| < \epsilon/(N-1) \leq 2\epsilon/N, \quad |S_{\mathcal{L}_N}(x,y)| < \epsilon/N.$$

It now follows from (17) that $|T_E(x,y)| < 7\epsilon/N$. Since $\epsilon > 0$ was arbitrary, we now conclude that the squarelike partial sum $T_E(x,y)$ is equal to $o(|E|^{-1})$ as $|E| \to \infty$. This also implies that $T(x,y)$ converges squarelikely to 0 everywhere. We have thus proved the assertion for the case of translation $\tau_{(1,0)}$. □

We can now complete the proof of the main theorem.

Proof of the main theorem. Let $S(x,y) = \sum_{m,n\in\mathbb{Z}} a_{m,n} e^{imx} e^{iny}$ be a 2-dimensional trigonometric series satisfying the hypothesis of the main theorem. Then, for each $(p,q) \in \mathbb{Z}^2$, the translate $\tau_{(p,q)} S(x,y)$, whose constant term is $a_{p,q}$, also satisfies the same hypothesis. Hence, by Corollary 8, $a_{p,q} = 0$. Therefore, all the coefficients $a_{m,n}$ are equal to 0. □

§4. Epilogue

The main result is stated under the restriction

$$S_E(x, y) = o(|E|^{-1}), \tag{18}$$

where E is a *squarelike* indexing set. We can compare (18) with the following condition about square partial sums :

$$\sum_{|m|<r} \sum_{|n|<r} a_{m,n} e^{imx} e^{iny} = o(1/r). \tag{19}$$

These two conditions are not equivalent. In fact, it is an open problem whether (19) can replace (18) in order to derive the main result. But, one can easily derive the following corollary from the main result.

COROLLAY 10. *Suppose that a 2-dimensional trigonometric series*

$$S(x, y) = \sum_{m,n \in \mathbb{Z}} a_{m,n} e^{imx} e^{iny}$$

satisfies

$$\sum_{|m|<r} \sum_{|n|<r} a_{m,n} e^{imx} e^{iny} = o(1/r)$$

and

$$\sum_{\max\{|m|,|n|\}=r} |a_{m,n}| = o(1/r).$$

Then all the coefficients $a_{m,n}$ *are 0.*

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