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DUAL OPERATOR ALGEBRAS; SUFFICIENT CONDITIONS FOR THE CLASSES $\mathbb{A}_{n,\aleph_0}$

YOUNG SOO JO[1], IL BONG JUNG[2,3] AND HUNG HWAN LEE [3]

Let $\mathcal{H}$ be a separable, infnite dimensional, complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. A *dual algebra* is a subalgebra of $\mathcal{L}(\mathcal{H})$ that contains the identity operator $I_{\mathcal{H}}$ and is closed in the ultraweak operator topology on $\mathcal{L}(\mathcal{H})$. This notion of dual algebras was introduced by S. Brown in [6], where he proved that every subnormal operator has a nontrivial invariant subspace. The theory of dual algebras is deeply related to the classes $\mathbb{A}_{m,n}$ which will be defined below and the study of the problem of solving systems of simultaneous equations in the predual of a singly generated dual algebra (see [1], [3] and [4]). This theory is applied to the study of invariant subspaces and dilation theory. In particular, in [7] Chevreau-Exner-Pearcy obtained some characterizations of operators in the class $\mathbb{A}_{1,\aleph_0}$. In addition, Exner-Jung [10] defined certain Hereditary properties concerning a minimal isometric dilation of T in $\mathbb{A}$ and obtained some characterizations for membership of the class $\mathbb{A}_{1,\aleph_0}$.

As a sequal study of them we define a certain index α_T of a contraction T and discuss relationship between the index α_T and the classes $\mathbb{A}_{n,\aleph_0}$ in this paper.

The notation and terminology employed here agree with those in [2], [4] and [18]. We recall nonetheless them for the convenience of the reader.

Suppose that $\mathcal{A}$ is a dual algebra in $\mathcal{L}(\mathcal{H})$. Let $\mathcal{C}_1 = \mathcal{C}_1(\mathcal{H})$ be the ideal of trace class operators in $\mathcal{L}(\mathcal{H})$ under the trace norm and let

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${}^{\perp}\mathcal{A}$ denote the preannihilator of $\mathcal{A}$ in $\mathcal{C}_1$. Let $\mathcal{Q}_{\mathcal{A}}$ denote the quotient space $\mathcal{C}_1/{}^{\perp}\mathcal{A}$. One knows that $\mathcal{A}$ is the dual space of $\mathcal{Q}_{\mathcal{A}}$ and that the duality is given by

$$(1) \qquad < T, [L] > = trace(TL), \; T \in \mathcal{A}, \; [L] \in \mathcal{Q}_{\mathcal{A}}.$$

Furthermore, the weak* topology that accrues to $\mathcal{A}$ by virtue of this duality coincides with the ultraweak operator topology on $\mathcal{A}$ (cf. [8]). For $T \in \mathcal{L}(\mathcal{H})$, let $\mathcal{A}_T$ denote the dual algebra generated by T. For vectors x and y in $\mathcal{H}$, we write, as usual, $x \otimes y$ for the rank one operator in $\mathcal{C}_1$ defined by

$$(x \otimes y)(u) = (u, y)x, \quad u \in \mathcal{H}.$$

We shall denote by $\mathbb{D}$ the open unit disc in the complex plane $\mathbb{C}$ and we write $\mathbb{T}$ for the boundary of $\mathbb{D}$. For $1 \leq p \leq \infty$ we denote the usual Lebesgue function space by $L^p = L^p(\mathbb{T})$. For $1 \leq p \leq \infty$ we denote by $H^p = H^p(\mathbb{T})$ the subspace of L^p consisting of those functions whose negative Fourier coefficients vanish. One knows that the preannihilator ${}^{\perp}(H^\infty)$ of H^∞ in L^1 is the subspace H_0^1 consisting of those functions g in H^1 for which analytic extension $\widetilde{g}$ to $\mathbb{D}$ satisfies $\widetilde{g}(0) = 0$. It is well known that H^∞ is the dual space of L^1/H_0^1.

Let us recall that any contraction T can be written as a direct sum $T = T_1 \oplus T_2$, where T_1 is a completely nonunitary contraction and T_2 is a unitary operator (cf. [18]). If T_2 is absolutely continuous or acts on the space (0), T will be called an *absolutely continuous contraction.*

The following provides a good relationship between the function space H^∞ and a singly generated dual algebra $\mathcal{A}_T$.

FOIAŞ-NAGY FUNCTIONAL CALCULUS. *Let T be an absolutely continuous contraction in $\mathcal{L}(\mathcal{H})$. Then there is an algebra homomorphism $\Phi_T : H^\infty \longrightarrow \mathcal{A}_T$ defined by $\Phi_T(f) = f(T)$ such that*

(a) $\Phi_T(1) = I_{\mathcal{H}}, \ \Phi_T(\xi) = T$,

(b) $\|\Phi_T(f)\| \leq \|f\|_\infty, \ f \in H^\infty$,

(c) *Φ_T is continuous if both H^∞ and $\mathcal{A}_T$ are given their weak* topologies,*

(d) *the range of Φ_T is weak* dense in $\mathcal{A}_T$,*

(e) *there exists a bounded, linear, one-to-one map $\phi_T : \mathcal{Q}_T \longrightarrow L^1/H_0^1$ such that $\phi_T^* = \Phi_T$, and*

(f) *if* Φ_T *is an isometry, then* Φ_T *is a weak** *homeomorphism of* H^∞ *onto* $\mathcal{A}_T$ *and* ϕ_T *is an isometry of* $\mathcal{Q}_T$ *onto* L^1/H^1_0.

Suppose that m and n are any cardinal numbers such that $1 \leq m, n \leq \aleph_0$. A dual algebra $\mathcal{A}$ will be said to have property $(\mathbb{A}_{m,n})$ if every $m \times n$ system of simultaneous equations of the form

$$[x_i \otimes y_j] = [L_{ij}], \ 0 \leq i < m, \ 0 \leq j < n,$$

where $\{[L_{ij}]\}_{\substack{0 \leq i < m \\ 0 \leq j < n}}$ is an arbitrary $m \times n$ array from $\mathcal{Q}_{\mathcal{A}}$, has a solution $\{x_i\}_{0 \leq i < m}$, $\{y_j\}_{0 \leq j < n}$ consisting of a pair of sequences of vectors from $\mathcal{H}$. For brevity, we shall denote $(\mathbb{A}_{n,n})$ by $(\mathbb{A}_n)$. The class $\mathbb{A}(\mathcal{H})$ consists of all those absolutely continuous contractions T in $\mathcal{L}(\mathcal{H})$ for which the functional calculus $\Phi_T : H^\infty \longrightarrow \mathcal{A}_T$ is an isometry. Furthermore, we denote by $\mathbb{A}_{m,n}(\mathcal{H})$ the set of all T in $\mathbb{A}(\mathcal{H})$ such that the algebra $\mathcal{A}_T$ has property $(\mathbb{A}_{m,n})$. We write simply $\mathbb{A}_{m,n}$ for $\mathbb{A}_{m,n}(\mathcal{H})$ unless we mention otherwise.

If $\mathcal{M}$ is a semi-invariant subspace for $T \in \mathcal{L}(\mathcal{H})$ (i.e., there exist invariant subspaces $\mathcal{N}_1$ and $\mathcal{N}_2$ for T with $\mathcal{N}_1 \supset \mathcal{N}_2$ such that $\mathcal{M} = \mathcal{N}_1 \ominus \mathcal{N}_2$), $T_{\mathcal{M}}$ denotes the *compression* of T to $\mathcal{M}$. In other words, $T_{\mathcal{M}} = P_{\mathcal{M}} T | \mathcal{M}$, where $P_{\mathcal{M}}$ is the orthogonal projection whose range is $\mathcal{M}$.

Throughout this paper, we write $\mathbb{N}$ for the set of natural numbers. For a Hilbert space $\mathcal{K}$ and any operators $T_i \in \mathcal{L}(\mathcal{K})$, $i = 1, 2$, we write $T_1 \cong T_2$ if T_1 is unitarily equivalent to T_2.

Recall that $T \in C_{\cdot 0}$ if $\|T^{*n} x\| \longrightarrow 0$ for any $x \in \mathcal{H}$. We say $T \in C_{0\cdot}$ if $T^* \in C_{\cdot 0}$. And we denote that $C_{00} = C_{0\cdot} \cap C_{\cdot 0}$.

Let T be a contraction operator in $\mathcal{L}(\mathcal{H})$ and we denote by $B_T \in \mathcal{L}(\mathcal{K}_+)$ a minimal isometric dilation of T, where

$$\mathcal{K}_+ = \bigvee_{n=0}^{\infty} B_T^n \mathcal{H}. \tag{2}$$

It follows from Wold decomposition theorem that

$$B_T = S_T \oplus R_T, \tag{3}$$

where $S_T \in \mathcal{L}(\mathcal{U}_T)$ is the unilateral shift part and $R_T \in \mathcal{L}(\mathcal{R}_T)$ is the residual part. Furthermore, it follows from (3) that $B_T^* = S_T^* \oplus R_T^*$ is a minimal coisometric extension of T^*. Let $U_T \in \mathcal{L}(\mathcal{K})$ be a minimal unitary dilation of T, where

$$\mathcal{K} = \bigvee_{n=-\infty}^{\infty} U_T^n \mathcal{H}. \tag{4}$$

Let $\mathcal{L}_T = \overline{(U_T - T)\mathcal{H}}$ and let $\mathcal{L}_{T^*} = \overline{(U_T^* - T^*)\mathcal{H}}$. Recall that $\dim\mathcal{L}_T = d_T$ and $\dim\mathcal{L}_{T^*} = d_{T^*}$, which are called *defect indices* of T and T^*, respectively.

Throughout this paper we denote a index by

$$\alpha_T = \dim(\mathcal{L}_T \cap \mathcal{R}_T). \tag{5}$$

PROPOSITION 1. *Suppose that T is a contraction on $\mathcal{H}$. Let $\mathcal{M}$ be an invariant subspace of T and let $\widetilde{T}$ be a restriction of T to $\mathcal{M}$. Then $\alpha_{\widetilde{T}} \leq \alpha_T$.*

Proof. Let $B_T \in \mathcal{L}(\mathcal{K}_+)$ be a minimal isometric dilation of T with $B_T = S_T \oplus R_T$, where $S_T \in \mathcal{L}(\mathcal{U}_T)$ is the shift part and $R_T \in \mathcal{L}(\mathcal{R}_T)$ is the residual part. Then B_T is the isometric dilation of $\widetilde{T}$. Hence there exists $\mathcal{K}'_+ \in \mathrm{Lat}(B_T)$ such that $B_T|\mathcal{K}'_+$ is a minimal isometric dilation of $\widetilde{T}$ with the decomposition

$$B_T|\mathcal{K}'_+ = S_{\widetilde{T}} \oplus R_{\widetilde{T}} \in \mathcal{U}_{\widetilde{T}} \oplus \mathcal{R}_{\widetilde{T}}.$$

We write $B_{\widetilde{T}} = B_T|\mathcal{K}'_+$. Since every minimal isometric dilations (or minimal unitary dilations) of a given contraction are unitary equivalent each other, it is sufficient to show that $\mathcal{L}_{\widetilde{T}} \subset \mathcal{L}_T$ and $\mathcal{R}_{\widetilde{T}} \subset \mathcal{R}_T$. Since one is obvious, it is sufficient to show that $\mathcal{R}_{\widetilde{T}} \subset \mathcal{R}_T$.

Let $x \in \mathcal{R}_{\widetilde{T}}$ and let $x = s \oplus r \in \mathcal{U}_T \oplus \mathcal{R}_T$. Since $B_{\widetilde{T}} = B_T|\mathcal{K}'_+$, we have

$$B_T^{*n} = \begin{pmatrix} B_{\widetilde{T}}^{*n} & 0 \\ A_n & * \end{pmatrix}$$

relative to a decomposition $\mathcal{K}'_+ \oplus (\mathcal{K} \ominus \mathcal{K}'_+)$, where A_n is some bounded operator from $\mathcal{K}'_+$ to $\mathcal{K} \ominus \mathcal{K}'_+$, for any $n \in \mathbb{N}$. Futhermore, we have

$$\begin{aligned} \|x\|^2 &\leq \|x\|^2 + \|A_n x\|^2 = \|R_{\widetilde{T}}^{*n} x\|^2 + \|A_n x\|^2 \\ &= \|B_{\widetilde{T}}^{*n} x \oplus A_n x\|^2 = \|B_T^{*n} x\|^2 \leq \|x\|^2. \end{aligned} \tag{6}$$

Hence $A_n x = 0$ for any $n \in \mathbb{N}$. This proves that

$$
\begin{aligned}
\|s\|^2 + \|r\|^2 &= \|x\|^2 = \|R_{\tilde{T}}^{*n} x\|^2 = \|B_{\tilde{T}}^{*n} x\|^2 \\
&= \|B_{\tilde{T}}^{*n} x\|^2 + \|A_n x\|^2 = \|B_{\tilde{T}}^{*n} x \oplus A_n x\|^2 \\
&= \|B_T^{*n} x\|^2 = \|S_T^{*n} s\|^2 + \|R_T^{*n} r\|^2 \\
&= \|S_T^{*n} s\|^2 + \|r\|^2.
\end{aligned} \tag{7}
$$

Letting $n \to \infty$ on the right side of (7), we have that $s = 0$. So $x \in \mathcal{R}_T$. Hence the proof is complete. □

The following is fundamental.

PROPOSITION 2. *Suppose that T_i is a contraction on $\mathcal{H}_i$ with $\alpha_{T_i} = r_i$, $i = 1, \cdots, n$. Let $\tilde{T} = \oplus_{i=1}^n T_i$. Then $\alpha_{\tilde{T}} = \sum_{i=1}^n r_i$.*

If $T \in C_{\cdot 0}$, then $\alpha_T = 0$. If T is an isometry, then $\alpha_T = 0$. In this paper we study completely nonunitary contraction operators with $\alpha_T \neq 0$. For example, if S is a unilateral shift operator of multiplicity $m \in \mathbb{N}$, then $\alpha_{S^*} = m$.

LEMMA 3. *Suppose that T is a completely nonunitary contraction on $\mathcal{H}$ with $\alpha_{T^*} = n \in \mathbb{N}$. Suppose that $\{u_k\}_{1 \le k \le n}$ is an orthonormal set in $\mathcal{L}_{T^*} \cap \mathcal{R}_{T^*}$. Let us denote*

$$M\Big(\bigvee_{k=1}^{n} \{u_k\}\Big) = \bigvee_{n=-\infty}^{\infty} R_{T^*}^n \Big(\bigvee_{k=1}^{n} \{u_k\}\Big) \tag{8}$$

and

$$M(u_k) = \bigvee_{n=-\infty}^{\infty} R_{T^*}^n u_k. \tag{9}$$

Then

$$M\Big(\bigvee_{k=1}^{n} \{u_k\}\Big) = \sum \oplus_{k=1}^{n} M(u_k). \tag{10}$$

Proof. Let u and v be any two vectors in $\{u_k\}_{1 \le k \le n}$. Since

$$\mathcal{L}_{T^*} = \overline{(U_{T^*} - T^*)\mathcal{H}},$$

there exist sequences $\{h_n\} \subset \mathcal{H}$ and $\{g_n\} \subset \mathcal{H}$ such that

$$u = \lim_n (U_{T^*} - T^*)h_n$$

and

$$v = \lim_n (U_{T^*} - T^*)g_n,$$

which implies that

$$\begin{aligned}
(R_{T^*}^k u, v) &= (R_{T^*}^k(\lim_n (U_{T^*} - T^*)h_n), \lim_n (U_{T^*} - T^*)g_n) \\
&= \lim_m \lim_n (U_{T^*}^k (U_{T^*} - T^*)h_m, (U_{T^*} - T^*)g_n) \\
&= \lim_m \lim_n ((U_{T^*}^k h_m, g_n) - (U_{T^*}^{k-1} T^* h_m, g_n) \\
&\qquad - (U_{T^*}^{k+1} h_m, T^* g_n) + (U_{T^*}^k T^* h_m, T^* g_n)) \\
&= \lim_m \lim_n ((T^{*k} h_m, g_n) - (T^{*k-1} T^* h_m, g_n) \\
&\qquad - (T^{*k+1} h_m, T^* g_n) + (T^{*k} T^* h_m, T^* g_n)) = 0,
\end{aligned} \tag{11}$$

$k = \pm 1, \pm 2, \cdots$. Note that $(u, v) = 0$. Hence we have $u_i \perp M(u_j)$, $i \neq j$ and the proof is complete. □

We write $M_{e^{it}}$ for the usual multiplication function on $L^2(\mathbb{T})$.

LEMMA 4. *Suppose that T is a completely nonunitary contraction on $\mathcal{H}$ with $\alpha_{T^*} = n \in \mathbb{N}$. Then*

$$R_{T^*} \cong \underbrace{M_{e^{it}} \oplus \cdots \oplus M_{e^{it}}}_{(n)} \oplus T_0$$

relative to some decomposition

$$\underbrace{L^2(\mathbb{T}) \oplus \cdots \oplus L^2(\mathbb{T})}_{(n)} \oplus \mathcal{K}.$$

Proof. The idea of this proof comes from [18, p.83-89]. Let $\mathcal{G} = \mathcal{L}_{T^*} \cap \mathcal{R}_{T^*}$. By Lemma 3, there exists a set of non-zero vectors $\{u_k\}_{k=1}^n$ in $\mathcal{G}$ such that

$$R_{T^*}|M(\mathcal{G}) \cong R_{T^*}|\sum \oplus_{k=1}^n M(u_k). \tag{12}$$

Now we consider a fixed u_k from the above. Let $\{E_t\}_{0\leq t\leq 2\pi}$ be the spectral family associated with $R_{T^*}|M(u_k)$. Note that E_t is absolutely continuous function of t. For $a \in \mathcal{K},\ m, n \in \mathbb{N}$, we have

$$
(13) \qquad \begin{aligned}
(R^m_{T^*}a, R^n_{T^*}a) &= \int_0^{2\pi} e^{i(m-n)t} d(E_t a, a) \\
&= \int_0^{2\pi} e^{i(m-n)t}\, p(t)\, dt,
\end{aligned}
$$

where

$$p(t) = \frac{d}{dt}(E_t a, a).$$

There exists a unitary Φ from $M(u_k)$ onto $L^2(\Omega_k)$, where

$$\Omega_k = \{t : t \in (0, 2\pi), p(t) > 0\}.$$

Since u_k is a nonzero *-cyclic vector for $R_{T^*}|M(u_k)$, we have $\Omega_k = (0, 2\pi)$, $k = 1, \cdots, n$. Since we can assume that $L^2(\mathbb{T})$ is identified with $L^2(\Omega_k)$,

$$R_T^*|M(u_k) \cong M_{e^{it}} \in \mathcal{L}(L^2(\mathbb{T})).$$

Hence we have

$$
(14) \qquad R_{T^*}|\left(\sum \oplus_{k=1}^n M(u_k)\right) \cong \underbrace{M_{e^{it}} \oplus \cdots \oplus M_{e^{it}}}_{(n)}
$$

and the proof is complete. □

The following lemma comes from [15] (or [17]).

LEMMA 5. *Let T be in $\mathbb{A}$ with R_T the unitary piece of its minimal coisometric extension and let R_T has multiplicity at least n on $\mathbb{T}$. Then $T \in \mathbb{A}_{n,\aleph_0}$.*

The above lemmas prove the following theorem.

THEOREM 6. *If $T \in \mathbb{A}$ with $\alpha_{T^*} = n$, then $T \in \mathbb{A}_{n,\aleph_0}$.*

Note that there are several characterizations of unitary operators in the classes $\mathbb{A}_{n,\aleph_0}$ in [11], [12] and [13]. Hence the following corollary provides several sufficient conditions for operators in the classes $\mathbb{A}_{n,\aleph_0}$.

COROLLARY 7. *Let T be a contraction operator on $\mathcal{H}$. Suppose that*

$$T \cong \begin{pmatrix} T_1 & * \\ 0 & T_2 \end{pmatrix}$$

relative to some decomposition, where T_1 is a completely nonunitary in $\mathbb{A}$ with $\alpha_{T_1^*} = m$ and T_2 is a unitary in $\mathbb{A}_n$. Then $T \in \mathbb{A}_{m+n,\aleph_0}$.

Proof. Since T_2 is unitary, $T_2 \in \mathbb{A}_{n,\aleph_0}$ and R_{T_2} has multiplicity at least n on $\mathbb{T}$ (cf. [13]). Since $R_T \cong R_{T_1} \oplus R_{T_2}$ (cf. [5]), the residual part of $T_1 \oplus T_2$ has multiplicity at least $m+n$ on $\mathbb{T}$. Hence by Lemma 5 we have $T \in \mathbb{A}_{\aleph_0,m+n}$. □

Let T be a contraction on $\mathcal{H}$. Recall that if $d_T < \infty$ and $d_{T^*} < \infty$, then T is a Fredholm operator and the Fredholm index $\text{ind}(T)$ is equal to $d_T - d_{T^*}$ (cf. [16]). Recall from [9, Theorem 2.4] that if $T \in \mathbb{A}_{n,\aleph_0} \cap C_{\cdot 0}$, then $d_{T^*} - d_T \geq n$.

PROPOSITION 8. *Let $T \in C_{\cdot 0}$ with $d_T < \infty$. Then $\text{ind}(T^*) \geq \alpha_{T^*}$.*

Proof. If $d_{T^*} = \infty$, since $\text{ind}(T^*) = \infty$, the proposition holds. Hence we can assume that $d_{T^*} < \infty$. By [9, Lemma 1.3] we have

$$R_T^* \cong B^{(n)}, \tag{15}$$

where B is the bilateral shift operator of multiplicity one and $n = \text{ind}T^*$. By Lemma 4, there exists an invariant subspace $\mathcal{M}$ of R_T^* such that

$$R_T^*|\mathcal{M} \cong B^{(r)},$$

where $r = \alpha_{T^*}$. Hence $n \geq r$. □

Recall that a completely nonunitary contraction $T \in \mathcal{L}(H)$ is said to be of *class* C_0 if there exists a non-zero function $u \in H^\infty(\mathbb{T})$ such that the functional calculus $u(T) = 0$. Let $S^{(n)}$ be the unilateral shift operator on a Hilbert space of multiplicity n. In [14, Theorem 1], they proved a generalization of the theorem that if $S^{(1)}$ is unitarily equivalent to an operator matrix form

$$\begin{pmatrix} S^{(1)} & * \\ 0 & E \end{pmatrix}$$

relative to a decomposition $\mathcal{M} \oplus \mathcal{N}$, then E is in a certain class C_0. We give a simple proof as following.

COROLLARY 9. *Suppose that $S^{(n)}$ is the unilateral shift operator of multiplicity n for a positive integer n and*

$$S^{(n)} \cong \begin{pmatrix} S^{(n)} & * \\ 0 & E \end{pmatrix}$$

relative to a decomposition $\mathcal{M} \oplus \mathcal{N}$. Then $E \in C_0$.

Proof. Since $S^{(n)} \in C_{\cdot 0}$, $E \in C_{\cdot 0}$. Let us consider the canonical decomposition

$$E = \begin{pmatrix} E_1 & * \\ 0 & E_2 \end{pmatrix}, \tag{16}$$

where $E_1 \in C_{00}(\mathcal{K}_1)$ and $E_2 \in C_{10}(\mathcal{K}_2)$. Since

$$S^{(n)} \cong \begin{pmatrix} S^{(n)} & * \\ 0 & E_1 \end{pmatrix}.$$

we have $d_{E_2^*} - d_{E_2} = 0$. Hence it follows from (15) that $E_2 \in C_{0\cdot}$, which implies that the canonical decomposition in (16) must be $E = E_1$. So $E \in C_{00}$. By [18, Theorem VI.5.2] $E \in C_0$. Hence the proof is complete. □

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Young Soo Jo
Keimyung University
Il Bong Jung & Hung Hwan Lee
Kyungpook National University