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A SINGULAR NONLINEAR BOUNDARY VALUE PROBLEM IN THE NONLINEAR CIRCULAR MEMBRANE UNDER NORMAL PRESSURE

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1. Introduction

The nonlinear boundary value problem

$$
\begin{aligned}
& y'' = f(x,y,y') = -\frac{3}{x}y' - \frac{g(x)}{y^2}, \qquad 0 < x < 1, \\
(1.1) \qquad & y'(0) = 0, \text{and either } (H): y(1) = \lambda > 0 \\
& \text{or } (S): y'(1) + (1-v)y(1) = 0, 1 - v > 0, \\
& g \in C[0,1], k \le g(x) \le K \text{ on } [0,1] \text{ for some } k, K > 0
\end{aligned}
$$

arises in the nonlinear circular membrane under normal pressure [2,3]. By a positive solution of (1.1) we mean a positive function $y(x) \in C^1[0,1] \cap C^2(0,1]$ that satisfies (1.1).

Previous existence and uniqueness theorems for (1.1) under the condition (H) or (S), using the iterative methods and the shooting method, have been given in [1-3]. Our emphasis in this paper is on treating (1.1) directly as a boundary value problem and on obtaining the existence of a positive solution via psitive solutions of perturbations of (1.1). The idea which is used here can be easily adapted to handle more general singular nonlinear problems.

We call a function $\beta \in C^2[0,1]$ a positive lower solution of (1.1) if

$$
\begin{aligned}
& \beta'' \ge f(x,\beta,\beta') \text{ on } (0,1), \beta > 0 \text{ on } [0,1], \beta'(0) \ge 0, \\
& (H): \beta(1) \le \lambda \ ((S): \beta'(1) + (1-v)\beta(1) \le 0).
\end{aligned}
$$

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The definition of a positive upper solution of (1.1) is given in a similar way. Similar definitions hold for a perturbation of (1.1) which will be given in section 2 or section 3.

In section 2, we consider the problem (1.1) under the condition (H). And the problem (1.1) under the condition (S) is considered in section 3.

2. Existence and uniqueness theorems under the condition (H)

For each positive number m, we consider the nonlinear boundary value problem

$$(2.1)_m \qquad \begin{aligned} & y'' = -\frac{3}{x+\frac{1}{m}}y' - \frac{g(x)}{y^2}, \qquad 0 < x < 1, \\ & y'(0) = 0, \text{ and } (H) : y(1) = \lambda > 0, \\ & g \in C[0,1], k \le g(x) \le K \text{ on } [0,1] \text{ for some } k, K > 0, \end{aligned}$$

which may be viewed as a perturbation of (1.1).

To prove the existence of a positive solution of (1.1), we establish the existence of a positive solution of $(2.1)_m$.

LEMMA 2.1. *$y_l = \lambda$ is a positive lower solution of $(2.1)_m$.*

Proof. It is clear that

$$y_l(x) > 0l'' \ge -\frac{g(x)}{\lambda^2}.$$

Thus y_l is a positive lower solution of $(2.1)_m$, which completes the proof.

LEMMA 2.2. *$y_{um} = -\frac{K}{8\lambda^2}\{(x+\frac{1}{m})^2 - (1+\frac{1}{m})^2\} + \lambda$ is a positive upper solution of $(2.1)_m$.*

Proof. It is clear that

$$y_{um}(x) > 0 \text{ on } (0,1), y_{um}(1) = \lambda, \text{ and } y'_{um}(0) \le 0.$$

Since $\lambda \le y_{um}(x)$ and $k \le g(x) \le K$ on [0,1],

$$\frac{g(x)}{y_{um}^2} \le \frac{K}{\lambda^2}$$

and so

$$y_{um}'' = -\frac{2K}{8\lambda^2} \le \frac{6K}{8\lambda^2} - \frac{g(x)}{y_{um}^2} = -\frac{3}{x+\frac{1}{m}}y_{um}' - \frac{g(x)}{y_{um}^2}.$$

Thus y_{um} is a positive upper solution of $(2.)_m$, which completes the proof.

The following result is clear from an application of Schauder's Fixed Point Theorem.

LEMMA 2.3. *There exists a solution y_m of $(2.1)_m, y_m \in C^2[0,1]$, such that*

$$y_l(x) \le y_m(x) \le y_{um}(x) \text{ on } [0,1],$$

where y_l and y_{um} are given in Lemma 2.1 and Lemma 2.2.

LEMMA 2.4. *If y_1 and y_2 are two positive solutions of $(2.1)_m$, then $y_1 \equiv y_2$.*

Proof. Let y_1 and y_2 be positive solutions of $(2.1)_m$. Then we obtain

$$\left(\left(x+\frac{1}{m}\right)^3(y_1'-y_2')\right)' = \frac{(x+\frac{1}{m})^3 g(x)}{y_1^2y_2^2}(y_1^2-y_2^2) \text{ on } (0,1). \tag{2.1}$$

If we multiply both sides of (2.1) by $(y_1 - y_2)$, then we obtain

$$\begin{aligned}&\left(\left(x+\frac{1}{m}\right)^3(y_1'-y_2')\right)'(y_1-y_2)\\ &\quad= \frac{(x+\frac{1}{m})^3 g(x)}{y_1^2y_2^2}(y_1+y_2)(y_1^2-y_2^2) \text{ on } (0,1).\end{aligned} \tag{2.2}$$

Therefore, if we integrate both sides of (2.2) from 0 to 1, then we obtain

$$\begin{aligned}0 &\le \int_0^1 \left(\left(x+\frac{1}{m}\right)^3(y_1'-y_2')\right)'(y_1-y_2)dx\\ &= -\int_0^1\left(x+\frac{1}{m}\right)^3(y_1'-y_2')^2dx \le 0.\end{aligned}$$

Thus we obtain $y_1' - y_2' = 0$ and $y_1 - y_2 =$ constan. Since $y_1(1) = y_2(1)$, we have $y_1 \equiv y_2$.

Note If y_m is a positive solution of $(2.1)_m$ for each $m > 0$, then $y_m(x) \geq \lambda$ on $[0,1]$ and hence all positive solutions of $(2.1)_m$, $m > 0$, are bounded below by λ on $[0,1]$.

LEMMA 2.5. *If y_m is a positive solution of $(2.1)_m$, then $y_m' < 0$ on $(0,1]$ and $y(x) > y(1) > 0$ on $[0,1)$.*

Proof. Since

$$\left(\left(x + \frac{1}{m}\right)^3 y_m'\right)' = -\frac{(x + \frac{1}{m})^3 g(x)}{y_m^2} < 0 \text{ on } (0,1),$$

we obtain that $(x + \frac{1}{m})^3 y_m'$ is strictly decreasing on [0,1], which implies that

$$y_m' < 0 \text{ on } (0,1] \text{ and } y(x) > y(1) = \lambda \text{ on } [0,1).$$

This completes the proof.

LEMMA 2.6. *If $m_1 \geq m_2$ and y_{m_1} and y_{m_2} are positive solutions of $(2.1)_{m_1}$ and $(2.1)_{m_2}$, respectively, then $y_{m_1}(x) \leq y_{m_2}(x)$ on $[0,1)$.*

Proof. It is clear from the fact that y_{m_2} is an upper solution of $(2.1)_{m_1}$.

THEOREM 2.7 (EXISTENCE). *If y_m is the positive solution of $(2.1)_m$ for each $m = 1, 2, 3, \cdots$,then the sequence $\{y_m\}$ converges to a positive solution y of (1.1).*

Proof. To prove this theorem, we prove the following steps:

Step 1 : $y_m \to y$ as $m \to \infty$.
Step 2 : $y \in C^1[0,1] \cap C^2(0,1]$.
Step 3: y is a solution of (2.1).

Step 1 : From Lemma 2.3 and Lemma 2.6, we know that the sequene $\{y_m\}$ is monotone decreasing in m and bounded below y λ. Therefore,

$$y_m \to y \text{ as } m \to \infty \text{ and } y(x) \geq \lambda \text{ on } [0,1].$$

Step 2 : If we integrate $((x + \frac{1}{m})^3 y_m')'$ from 0 to x, then we have

$$\left(x + \frac{1}{m}\right)^3 y_m'(x) = \int_0^x -\frac{(\xi + \frac{1}{m})^3 g(\xi)}{y_m^2(\xi)} d\xi \tag{2.3}$$

and

$$\text{(2.4)} \qquad -y_m'(x) = \frac{1}{(x+\frac{1}{m})^3}\int_0^x \frac{(\xi+\frac{1}{m})^3 g(\xi)}{y_m^2(\xi)}d\xi.$$

If we integrate both sides of (2.4) from 1 to x, then we obtain

$$\text{(2.5)} \qquad \begin{aligned} y_m(x) =&\lambda + \int_x^1 \frac{1}{(s+\frac{1}{m})^3}\int_0^s \frac{(\xi+\frac{1}{m})^3 g(\xi)}{y_m^2(\xi)}d\xi ds \\ =&\lambda - \frac{1}{2(1+\frac{1}{m})^2}\int_0^1 \frac{(\xi+\frac{1}{m})^3 g(\xi)}{y_m^2(\xi)}d\xi \\ &+ \frac{1}{2(x+\frac{1}{m})^2}\int_0^x \frac{(\xi+\frac{1}{m})^3 g(\xi)}{y_m^2(\xi)}d\xi \\ &+ \frac{1}{2}\int_x^1 \frac{(\xi+\frac{1}{m})g(\xi)}{y_m^2(\xi)}d\xi \end{aligned}$$

and

$$\text{(2.6)} \qquad \begin{aligned} y_m(0) =&\lambda - \frac{1}{2(1+\frac{1}{m})^2}\int_0^1 \frac{\xi+\frac{1}{m})^3 g(\xi)}{y_m^2(\xi)}d\xi \\ &+ \frac{1}{2}\int_0^1 \frac{\xi+\frac{1}{m})g(\xi)}{y_m^2(\xi)}d\xi. \end{aligned}$$

If we let $m \to \infty$ in both sides of (2.5) and (2.6), then by Lebesque's Dominated Convergence Theorem, we obtain

$$\text{(2.7)} \qquad y(0) = \lambda - \frac{1}{2}\int_0^1 \frac{\xi^3 g(\xi)}{y^2(\xi)}d\xi + \frac{1}{2}\int_0^1 \frac{\xi g(\xi)}{y^2(\xi)}d\xi$$

and

$$\text{(2.8)} \qquad \begin{aligned} y(x) =&\lambda - \frac{1}{2}\int_0^1 \frac{\xi^3 g(\xi)}{y^2(\xi)}d\xi + \frac{1}{2x^2}\int_0^x \frac{\xi^3 g(\xi)}{y^2(\xi)}d\xi \\ &+ \frac{1}{2}\int_x^1 \frac{\xi g(\xi)}{y^2(\xi)}d\xi \quad \text{on } (0,1], \end{aligned}$$

which implies $y \in C^2(0,1]$. Since the second term of the right side of (2.8) converges to 0 as x approaches $0, y$ is continuous at 0. From (2.7) and (2.8), we obtain

$$\begin{aligned}(2.9)\quad &\lim_{x\to 0+}\frac{y(x)-y(0)}{x}\\ &=\lim_{x\to 0+}\left(\frac{1}{2x^3}\int_0^x \frac{\xi^3 g(\xi)}{y^2(\xi)}d\xi-\frac{1}{2x}\int_0^x \frac{\xi g(\xi)}{y^2(\xi)}d\xi\right)=0,\end{aligned}$$

which implies $y'(0)=0$. If we take the first derivative of both sides of (2.8), we have

$$(2.10)\qquad y'(x)=-\frac{1}{x^3}\int_0^x \frac{\xi^3 g(\xi)}{y^2(\xi)}d\xi \text{ on } (0,1]$$

and so

$$\lim_{x\to 0+} y'(x)=\lim_{x\to 0+}-\frac{1}{x^3}\int_0^x \frac{\xi^3 g(\xi)}{y^2(\xi)}d\xi=0,$$

which implies $y\in C^1[0,1]\cap C^2(0,1]$.

Step 3 : It is clear from (2.8) and (2.9) that $y(1)=\lambda$ and $y'(0)=0$. If we take the derivative of both sides of (2.10), then we get

$$y''(x)=-\frac{x^3\frac{g(x)}{y^2(x)}x^3-3x^2\int_0^x \frac{\xi^3 g(\xi)}{y^2(\xi)}d\xi}{x^6}=-\frac{g(x)}{y^2(x)}-\frac{3}{x}y'(x),$$

which implies that y is a solution of (1.1). This completes the proof.

THEOREM 2.8 (UNIQUENESS). *Assume that y_1 and y_2 are positive solutions of (1.1). Then $y_1 \equiv y_2$.*

Proof. The proof of this theorem is similar to that of Lemma 2.4.

3. Existence and uniqueness theorems under the condition (S)

For each positive number m, we consider the nonlinear boundary value problem

$$(3.1)_m\qquad \begin{aligned} &y''=-\frac{3}{x+\frac{1}{m}}y'-\frac{g(x)}{y^2},\quad 0<x<1,\\ &y'(0)=0,\ \text{and } (S): y'(1)+(1-v)y(1)=0,\quad 1-v>0,\\ &g\in C[0,1],\quad k\le g(x)\le K \text{ on } [0,1] \text{ for some } k,K>0,\end{aligned}$$

which may also be viewed as a perturbation of (1.1).

LEMMA 3.1. *$y_l = -(\frac{k(1-v)^2}{8(3-v)^2})^{\frac{1}{3}}\cdot(x^2 - \frac{3-v}{1-v})$ is a positive lower solution of $(3.1)_m$.*

Proof. It is clear that

$$y_l(x) > 0 \text{ on } (0,1),$$

$$y_l'(x) = -2x\left(\frac{x(1-v)^2}{8(3-v)^2}\right)^{\frac{1}{3}},$$

$$y_l'(0) = 0, y_l'(1) + (1-v)y_l(1) = 0,$$

and

$$y_l''(x) = -2\left(\frac{k(1-v)^2}{8(3-v)^2}\right)^{\frac{1}{3}}.$$

Since $y_l(x) \le (\frac{k(1-v)^2}{8(3-v)^2})^{\frac{1}{3}}\cdot(\frac{3-v}{1-v})$ and $k \le g(x)$ on [0,1], we obtain

$$\frac{g(x)}{y_l^2} \ge \frac{k}{(\frac{k(1-v)^2}{8(3-v)^2})^{\frac{2}{3}}(\frac{3-v}{1-v})^2} = 8\left(\frac{k(1-v)^2}{8(3-v)^2}\right)^{\frac{1}{3}}.$$

So we have

$$\begin{aligned} y_l'' &= -2\left(\frac{k(1-v)^2}{8(3-v)^2}\right)^{\frac{1}{3}} \\ &\ge -2\left(\frac{k(1-v)^2}{8(3-v)^2}\right)^{\frac{1}{3}} - 6\left(\frac{k(1-v)^2}{8(3-v)^2}\right)^{\frac{1}{3}}\left(1 - \frac{x}{x+\frac{1}{m}}\right) \\ &\ge -\frac{3}{x+\frac{1}{m}}y_l' - \frac{g(x)}{y_l^2}. \end{aligned}$$

Thus y_l is a positive lower solution of $(3.1)_m$, which completes the proof.

LEMMA 3.2. *$y_{um} = -\left(\frac{K(1-v)^2}{32}\right)^{\frac{1}{3}}\cdot((x+\frac{1}{m})^2 - \frac{3-v}{1-v}(1+\frac{1}{m})^2)$ is a positive upper solution of $(3.1)_m$.*

Proof. It is clear that

$$y_{um}(x) > 0 \text{ on } (0,1),$$

$$y_{um}'(x) = -2\left(x+\frac{1}{m}\right)\left(\frac{(1-v)^2}{32}\right)^{\frac{1}{3}},$$

$$y_{um}'(0) \le 0, y_{um}'(1) + (1-v)y_{um}(1) \ge 0,$$

and

$$y''_{um}(x) = -2\left(\frac{K(1-v)^2}{32}\right)^{\frac{1}{3}}.$$

Since $y_{um}(x) \geq (\frac{K(1-v)^2}{32})^{\frac{1}{3}} \cdot (1+\frac{1}{m})^2(1-\frac{3-v}{1-v})$ and $K \geq g(x)$ on $[0,1]$, we have

$$\frac{g(x)}{y_{um}^2} \leq \frac{K}{(\frac{K(1-v)^2}{32})^{\frac{2}{3}} \cdot (1+\frac{1}{m})^4(\frac{2}{1-v})^2} \leq 8\left(\frac{K(1-v)^2}{32}\right)^{\frac{1}{3}}.$$

Therefore we obtain

$$\begin{aligned} y''_{um}(x) &= -2\left(\frac{K(1-v)^2}{32}\right)^{\frac{1}{3}} \\ &\leq 6\left(\frac{K(1-v)^2}{32}\right)^{\frac{1}{3}} - 8\left(\frac{K(1-v)^2}{32}\right)^{\frac{1}{3}} \\ &\leq -\frac{3}{x+\frac{1}{m}}y'_{um} - \frac{g(x)}{y_{um}^2}. \end{aligned}$$

Thus y_{um} is a positive upper solution of $(3.1)_m$. This completes the proof.

The following result is clear from an application of Schauder's Fixed Point Theorem.

LEMMA 3.3. *There exists a positive solution y_m of $(3.1)_m, y_m \in C^2[0,1]$, such that*

$$y_l(x) \leq y_m(x) \leq y_{um}(x) \text{ on } [0,1],$$

where y_l and y_{um} are given in Lemma 3.1 and Lemma 3.2.

LEMMA 3.4. *If y_1 and y_2 are two positive solutions of $(3.1)_m$, then $y_1 \equiv y_2$.*

Proof. Let y_1 and y_2 be positive solutions of $(3.1)_m$. Then we obtain

$$(3.1) \qquad ((x+\frac{1}{m})^3(y'_1-y'_2))' = \frac{(x+\frac{1}{m})^3 g(x)}{y_2^2 y_2^2}(y_1^2-y_2^2) \text{ on } (0,1).$$

If we multiply both sides of (3.1) by $(y_1 - y_2)$, then we obtain

$$\begin{aligned}(3.2)\qquad &\left(\left(x+\frac{1}{m}\right)^3(y_1'-y_2')\right)'(y_1-y_2)\\ &=\frac{(x+\frac{1}{m})^3 g(x)}{y_1^2y_2^2}(y_1+y_2)(y_1-y_2)^2\geq 0 \text{ on } (0,1).\end{aligned}$$

Therefore, if we integrate both sides of (3.2) from 0 to 1, then we obtain

$$\begin{aligned}0&\leq\int_0^1\left(\left(x+\frac{1}{m}\right)^3(y_1'-y_2')\right)'(y_1-y_2)dx\\ &\leq\left(1+\frac{1}{m}\right)^3(y_1'(1)-y_2'(1))(y_1(1)-y_2(1)))\\ &\quad-\int_0^1\left(x+\frac{1}{m}\right)^3(y_1'-y_2')^2dx\\ &\leq-\left(1+\frac{1}{m}\right)^3(1-v)(y_1(1)-y_2(1))^2-\int_0^1\left(x+\frac{1}{m}\right)^3(y_1'-y_2')^2dx\end{aligned}$$

and so

$$\left(1+\frac{1}{m}\right)^3(1-v)(y_1(1)-y_2(1))^2+\int_0^1\left(x+\frac{1}{m}\right)^3(y_1'-y_2')^2dx=0,$$

which implies that

$$y_1(1)-y_2(1)=0 \text{ and } y1'-y_2'\equiv 0 \text{ on } [0,1].$$

Thus $y_1\equiv y_2$ on $[0,1]$, which completes the proof.

Note. If y_m is a positive solution of $(3.1)_m$ for each $m>0$, then

$$y_m(x)\geq\left(\frac{k(1-v)^2}{8(3-v)^2}\right)^{\frac{1}{3}}\cdot\frac{2}{1-v} \text{ on } [0,1]$$

and so all positive solutions of $(3.1)_m$, $m>0$, are bounded below by some positive number on $[0,1]$.

LEMMA 3.5. *If y is a positive solution of $(3.1)_m$, then $y'(x) < 0$ on $(0,1]$ and $y(x) > y(1) > 0$ on $[0,1)$.*

Proof. Since

$$\left(\left(x+\frac{1}{m}\right)^3 y_m'\right)' = -\frac{(x+\frac{1}{m})^3 g(x)}{y_m^2} < 0 \text{ on } (0,1),$$

we obtain that $(x+\frac{1}{m})^3 y_m'$ is strictly decreasing on [0,1], whic h implies that

$$y_m' < 0 \text{ on } (0,1] \text{ and } y(x) > y(1) \text{ on } [0,1).$$

Since $y'(1) < 0$ and $y(1) = -\frac{1}{1-v}y'(1), y(1) > 0$. Thus $(x) > y(1) > 0$ on $[0,1)$, which completes the proof.

LEMMA 3.6. *If $m_1 \geq m_2$ and y_{m_1} and y_{m_2} are positive solutions of $(3.1)_{m_1}$ and $(3.1)_{m_2}$, respectively, then $y_{m_1}(x) \leq y_{m_2}(x)$ on $[0,1]$.*

Proof. It is clear from the fact that y_{m_2} is an upper solution of $(3.1)_{m_1}$.

THEOREM 3.7 (EXISTENCE). *If y_m is a positive solution of $(3.1)_m$ for each $m = 1,2,3,\cdots$, then the sequence $\{y_m\}$ converges to a positive solution y of (1.1).*

Proof. To prove this theorem, we prove the following steps:

Step 1: $y_m \to y$ as $m \to \infty$.
Step 2: $y \in C^1[0,1] \cap C^2(0,1]$.
Step 3: y is a solution of (1.1).

Step 1: From Lemma 3.3 and Lemma 3.6, we know that the sequence $\{y_m\}$ is monotone decreasing in m and bounded below by some positive number. Therefore,

$$y_m \to y \text{ as } m \to \infty \text{ and } y(x) > 0 \text{ on } [0,1].$$

Step 2: If we integrate $((x+\frac{1}{m})^3 y_m')'$ from 0 to x, then we have

$$\left(x+\frac{1}{m}\right)^3 y_m'(x) = \int_0^x -\frac{(\xi+\frac{1}{m})^3 g(\xi)}{y_m^2(\xi)} d\xi \tag{3.3}$$

and

$$-y_m'(x) = \frac{1}{(x+\frac{1}{m})^3}\int_0^x \frac{(\xi+\frac{1}{m})^3 g(\xi)}{y_m^2(\xi)}d\xi. \tag{3.4}$$

If we integrate both sides of (3.4) from 1 to x, then we obtain

$$\begin{aligned} y_m(x) =& y_m(1) + \int_x^1 \frac{1}{(s+\frac{1}{m})^3}\int_0^s \frac{(\xi+\frac{1}{m})^3 g(\xi)}{y_m^2(\xi)}d\xi ds \\ =& y_m(1) - \frac{1}{2(1+\frac{1}{m})^2}\int_0^1 \frac{(\xi+\frac{1}{m})^3 g(\xi)}{y_m^2(\xi)}d\xi \\ &+ \frac{1}{2(x+\frac{1}{m})^2}\int_0^x \frac{(\xi+\frac{1}{m})^3 g(\xi)}{y_m^2(\xi)}d\xi \\ &+ \frac{1}{2}\int_x^1 \frac{(\xi+\frac{1}{m}) g(\xi)}{y_m^2(\xi)}d\xi \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} y_m(0) = y_m(1) &- \frac{1}{2(1+\frac{1}{m})^2}\int_0^1 \frac{(\xi+\frac{1}{m})^3 g(\xi)}{y_m^2(\xi)}d\xi \\ &+ \frac{1}{2}\int_0^1 \frac{(\xi+\frac{1}{m}) g(\xi)}{y_m^2(\xi)}d\xi. \end{aligned} \tag{3.6}$$

If we let $m \to \infty$ in both sides of (3.5) and (3.6), then by Lebesque's Dominated Convergence Theorem, we obtain

$$y(0) = y(1) - \frac{1}{2}\int_0^{\frac{\xi^3 g(\xi)}{y^2(\xi)}} d\xi + \frac{1}{2}\int_0^1 \frac{\xi g(\xi)}{y^2(\xi)}d\xi \tag{3.7}$$

and

$$\begin{aligned} y(x) =& y(1) - \frac{1}{2}\int_0^1 \frac{\xi^3 g(\xi)}{y^2(\xi)}d\xi + \frac{1}{2x^2}\int_0^x \frac{\xi^3 g(\xi)}{y^2(\xi)}d\xi \\ &+ \frac{1}{2}\int_x^1 \frac{\xi g(\xi)}{y^2(\xi)}d\xi \text{ on } (0,1], \end{aligned} \tag{3.8}$$

which implies $y \in C^2(0,1]$. Since the second term of the right side of (3.8) converges to 0 as x approaches 0, y is continuous at 0. From (3.7) and (3.8), we obtain

$$
\begin{aligned}
y'(0) &= \lim_{x \to 0+} \frac{y(x) - y(0)}{x} \\
&= \lim_{x \to 0+} \left(\frac{1}{2x^3} \int_0^x \frac{\xi^3 g(\xi)}{y^2(\xi)} d\xi - \frac{1}{2x} \int_0^x \frac{\xi g(\xi)}{y^2(\xi)} d\xi \right) \\
&= 0.
\end{aligned} \tag{3.9}
$$

If we take the first derivative of both sides of (3.8), we obtain

$$
y'(x) = -\frac{1}{x^3} \int_0^x \frac{\xi^3 g(\xi)}{y^2 \xi)} d\xi \text{ on } (0,1] \tag{3.10}
$$

and so

$$
\lim_{x \to 0+} y'(x) = \lim_{x \to 0+} -\frac{}{x^3} \int_0^x \frac{\xi^3 g(\ xi)}{y^2(\xi)} d\xi = 0,
$$

which implies $y \in C^1[0,1] \cap C^2(0,1]$.

Step 3: It is clear from (3.9) and (3.10) that

$$
y'(0) = 0 \text{ and } y'(1) = -\int_0^1 \frac{\xi^3 g(\xi)}{y^2(\xi)} d\xi.
$$

On the other hand, from (3.4), we obtain

$$
\begin{aligned}
y'(1) = -\int_0^1 \frac{\xi^3 g(\xi)}{y^2(\xi)} d\xi &= \lim_{m \to \infty} \left(-\frac{1}{(1+\frac{1}{m})^3} \int_0^1 \frac{(\xi + \frac{1}{m})^3 g(\xi)}{y_m^2(\xi)} d\xi \right) \\
= \lim_{m \to \infty} y'_m(1) &= -(1-v)y(1).
\end{aligned}
$$

If we take the derivative of both sides of (3.10), then we get

$$
y''(x) = -\frac{x^3 \frac{g(x)}{y^2(x)} x^3 - 3x^2 \int_0^x \frac{\xi^3 g(\xi)}{y^2(\xi)} d\xi}{x^6} = -\frac{g(x)}{y^2(x)} - \frac{3}{y}'(x),
$$

which implies that y is a solution of (1.1). This completes the proof.

THEOREM 3.8 (UNIQUENESS). *Assume that y_1 and y_2 are positive solutionsof (1.1). Then $y_1 \equiv y_2$.*

Proof. The proof of this theorem is similar to that of Lemma 3.4.

References

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